11 Three Dimensional Geometry

Fastrack Revision

- If a line makes angles α, β, γ from the coordinate axes respectively, then its direction cosines l, m, n are as follows:

$$l = \cos\alpha, m = \cos\beta, n = \cos\gamma$$

- For the line joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2):
  1. Direction ratios $= x_2 - x_1, y_2 - y_1, z_2 - z_1$
  2. Direction cosines $= \frac{x_2 - x_1}{AB}, \frac{y_2 - y_1}{AB}, \frac{z_2 - z_1}{AB}$

  where, $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

- If the direction ratios of a line are a, b and c, then its direction cosines are:

$$l = \pm\frac{a}{\sqrt{a^2 + b^2 + c^2}}, m = \pm\frac{b}{\sqrt{a^2 + b^2 + c^2}}, n = \pm\frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

Angle between Two Lines

- If the direction cosines of two lines are l_1, m_1, n_1 and l_2, m_2, n_2 then the angle θ between them is given by the following formula:

$$\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$$

  1. Lines will be perpendicular, if $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$.
  2. Lines will be parallel, if $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$.

- If the direction ratios of two lines are a_1, b_1, c_1 and a_2, b_2, c_2 then the angle θ between them is given by the following formula:

$$\cos\theta = \left|\frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{(a_1^2 + b_1^2 + c_1^2)} \cdot \sqrt{(a_2^2 + b_2^2 + c_2^2)}}\right|$$

  1. Lines will be perpendicular, if $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$.
  2. Lines will be parallel, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

- **Area of Triangle:** If the vertices of a triangle are $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$, then

  Area of Δ ABC $= \sqrt{\Delta_x^2 + \Delta_y^2 + \Delta_z^2}$

  where, $\Delta_x = \frac{1}{2}\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}$, $\Delta_y = \frac{1}{2}\begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix}$

  and $\Delta_z = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

Vector Equation of a Line

- Equation of a line passing through point $\vec{a}$ and parallel to vector $\vec{b}$:

$$\vec{r} = \vec{a} + t\vec{b} \quad \text{or} \quad \vec{r} \times \vec{b} = \vec{a} \times \vec{b}$$

- Equation of a line passing through the point $\vec{a}$ and perpendicular to two non-parallel vectors $\vec{c}$ and $\vec{d}$ is:

$$\vec{r} = \vec{a} + t(\vec{c} \times \vec{d}) \quad \text{or} \quad \vec{r} \times (\vec{c} \times \vec{d}) = \vec{a} \times (\vec{c} \times \vec{d})$$

Cartesian Equation of a Line

- Equation of a line passing through point (x_1, y_1, z_1) whose direction ratios are a, b, c:

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}.$$

Angle between Two Lines in Vector Form

- Let vector equations of two lines are

$$\vec{r} = \vec{a_1} + t\vec{b_1} \text{ and } \vec{r} = \vec{a_2} + t\vec{b_2}$$

  If θ be the angle between the lines, then

$$\cos\theta = \left|\frac{\vec{b_1} \cdot \vec{b_2}}{|\vec{b_1}||\vec{b_2}|}\right|$$

- Lines will be perpendicular, if $\vec{b_1} \cdot \vec{b_2} = 0$
- Lines will be parallel, if $\vec{b_1} = k\vec{b_2}$, where k is a scalar quantity.

Collinearity of Three Given Points

- Three points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) are collinear, if $\frac{x_3 - x_1}{x_2 - x_1} = \frac{y_3 - y_1}{y_2 - y_1} = \frac{z_3 - z_1}{z_2 - z_1}$.
- When three points A, B, C with position vectors $\vec{a}, \vec{b}, \vec{c}$ respectively are collinear if and only if there exists scalars μ_1, μ_2, μ_3 not all zero, such that $\mu_1 \vec{a} + \mu_2 \vec{b} + \mu_3 \vec{c} = 0$ and $\mu_1 + \mu_2 + \mu_3 = 0$.

Shortest Distance (SD) between Two Skew-lines

- Two lines in space which are neither parallel nor intersecting is said to be skew lines.
- **Vector Form**

$$\vec{r} = \vec{r_1} + \lambda\vec{u} \quad \text{and} \quad \vec{r} = \vec{r_2} + \mu\vec{v}$$

$$SD = \left|\frac{(\vec{r_2} - \vec{r_1}) \cdot (\vec{u} \times \vec{v})}{|\vec{u} \times \vec{v}|}\right|$$

  If $[(\vec{r_2} - \vec{r_1}) \cdot (\vec{u} \times \vec{v})] = 0$, then lines will intersect.

- **Cartesian Form**

  Shortest distance between the lines $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ and $\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$ is given by:

$$SD = \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}}$$

If $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$ then lines will intersect.

Distance between Parallel Lines

▶ If $\vec{r} = \vec{r_1} + \lambda \vec{u}$ and $\vec{r} = \vec{r_2} + \mu \vec{u}$ are two parallel lines, then

$$d = \frac{|\vec{u} \times (\vec{r_2} - \vec{r_1})|}{|\vec{u}|}.$$

Practice Exercise

Multiple Choice Questions

Q 1. If a line makes angles of 90°, 135° and 45° with the X, Y and Z-axes respectively, then its direction cosines are: *(CBSE 2023)*

a. $0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$
b. $-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}$
c. $\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}$
d. $0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$

Q 2. P is a point on the line segment joining the points (3, 2, −1) and (6, 2, −2). If x-coordinate of P is 5, then its y-coordinate is: *(NCERT EXEMPLAR)*

a. 2
b. 1
c. −1
d. −2

Q 3. If the direction cosines of a line are $<\frac{1}{c}, \frac{1}{c}, \frac{1}{c}>$, then *(CBSE SQP 2023-24)*

a. $0 < c < 1$
b. $c > 2$
c. $c = \pm\sqrt{2}$
d. $c = \pm\sqrt{3}$

Q 4. Distance of the point (p, q, r) from Y-axis is: *(CBSE 2023)*

a. q
b. $|q|$
c. $|q| + |r|$
d. $\sqrt{p^2 + r^2}$

Q 5. Find the direction cosines of the line joining the points A(0, 7, 10) and B(−1, 6, 6).

a. $\frac{-1}{3\sqrt{2}}, \frac{-1}{3\sqrt{2}}, \frac{2}{3\sqrt{2}}$
b. $\frac{1}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}, \frac{4}{3\sqrt{2}}$
c. $\frac{1}{3}, \frac{-1}{3}, \frac{4}{3}$
d. None of these

Q 6. If P(2, 3, 4), Q(−1, −2, 1) and R(5, 8, 7) are three points, then:

a. direction ratios of PQ and QR are not in proportion
b. P, Q, R lie on the same line
c. P, Q, R do not lie on the same line
d. None of the above

Q 7. The point $(x, y, 0)$ on the xy-plane divides the line segment joining the points (1, 2, 3) and (3, 2, 1) in the ratio: *(CBSE 2023)*

a. 1 : 2 internally
b. 2 : 1 internally
c. 3 : 1 internally
d. 3 : 1 externally

Q 8. The equation of a line passing through the point (−3, 2, −4) and equally inclined to the axes are:

a. $x - 3 = y + 2 = z - 4$
b. $x + 3 = y - 2 = z + 4$
c. $\frac{x+3}{1} = \frac{y-2}{2} = \frac{z+4}{3}$
d. None of these

Q 9. If vector equation of the line $\frac{x-2}{2} = \frac{2y-5}{-3} = z + 1$ is $\vec{r} = \left(2\hat{i} + \frac{5}{2}\hat{j} - \hat{k}\right) + \lambda\left(2\hat{i} - \frac{3}{2}\hat{j} + p\hat{k}\right)$, then p is equal to:

a. 0
b. 1
c. 2
d. 3

Q 10. Vector equation of the line $6x - 3 = 3y + 4 = 2z - 2$ is:

a. $\vec{r} = \hat{i} - \hat{j} + \hat{k} + \lambda(6\hat{i} + \hat{j} + \hat{k})$
b. $\vec{r} = 6\hat{i} + 3\hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} - 2\hat{k})$
c. $\vec{r} = \frac{1}{2}\hat{i} - \frac{4}{3}\hat{j} + \hat{k} + \lambda\left(\frac{1}{6}\hat{i} + \frac{1}{3}\hat{j} + \frac{1}{2}\hat{k}\right)$
d. None of the above

Q 11. The lines $\frac{x-2}{1} = \frac{y-3}{1} = \frac{4-z}{k}$ and $\frac{x-1}{k} = \frac{y-4}{2} = \frac{z-5}{-2}$ are mutually perpendicular, if the value of k is: *(CBSE 2020)*

a. $-\frac{2}{3}$
b. $\frac{2}{3}$
c. −2
d. 2

Q 12. If the straight lines $\frac{x-1}{k} = \frac{y-2}{2} = \frac{z-3}{3}$ and $\frac{x-2}{3} = \frac{y-3}{-5} = \frac{z-1}{2}$ intersect at a point, then the integer k is equal to:

a. −5
b. 5
c. 2
d. None of these

Q 13. The lines $\vec{r} = \hat{i} + \hat{j} - \hat{k} + \lambda(2\hat{i} + 3\hat{j} - 6\hat{k})$ and $\vec{r} = 2\hat{i} - \hat{j} - \hat{k} + \mu(6\hat{i} + 9\hat{j} - 18\hat{k})$; (where λ & μ are scalars) are: *(CBSE SQP 2023-24)*

a. coincident
b. skew
c. intersecting
d. parallel

Q 14. The line of shortest distance between two skew-lines is: *(CBSE 2020)*

a. parallel to both the lines
b. perpendicular to both the lines
c. coincide with both the lines
d. None of the above

Q 15. **The equation of straight line passing through the point (a, b, c) and parallel to Z-axis is:**

a. $\frac{x-a}{1}=\frac{y-b}{1}=\frac{z-c}{0}$ b. $\frac{x-a}{0}=\frac{y-b}{1}=\frac{z-c}{1}$

c. $\frac{x-a}{1}=\frac{y-b}{0}=\frac{z-c}{0}$ d. $\frac{x-a}{0}=\frac{y-b}{0}=\frac{z-c}{1}$

Q 16. **The angle between a line whose direction ratios are in the ratio 2 : 2 : 1 and a line joining (3, 1, 4) to (7, 2, 12), is:**

a. $\cos^{-1}(2/3)$ b. $\cos^{-1}(-2/3)$

c. $\tan^{-1}(2/3)$ d. None of these

Q 17. **The acute angle between the lines**

$\vec{r}=(4\hat{i}-\hat{j})+s(2\hat{i}+\hat{j}-3\hat{k})$

and $\vec{r}=(\hat{i}-\hat{j}+2\hat{k})+t(\hat{i}-3\hat{j}+2\hat{k})$ **is:**

a. $\frac{3\pi}{2}$ b. $\frac{\pi}{3}$ c. $\frac{2\pi}{3}$ d. $\frac{\pi}{6}$

Q 18. **The angle between the lines $2x=3y=-z$ and $6x=-y=-4z$ is:** *(CBSE 2023)*

a. 0° b. 30°

c. 45° d. 90°

Q 19. **Equation of a line passing through (1, 2, −3) and parallel to the line $\frac{x-2}{1}=\frac{y+1}{3}=\frac{z-1}{4}$ is:**

a. $\frac{x-1}{1}=\frac{y-2}{3}=\frac{z+3}{4}$ b. $\frac{x-2}{1}=\frac{y+1}{2}=\frac{z-1}{-3}$

c. $\frac{x-1}{1}=\frac{y-3}{2}=\frac{z-4}{-3}$ d. None of these

Q 20. **If the line joining (2, 3, −1) and (3, 5, −3) is perpendicular to the line joining (1, 2, 3) and $(3, 5, \lambda)$, then $\lambda=$**

a. −3 b. 2 c. 5 d. 7

Q 21. **The shortest distance between the lines $x=y=z$ and $x=1-y=\frac{z}{0}$ is:**

a. $\frac{1}{2}$ b. $\frac{1}{\sqrt{2}}$ c. $\frac{1}{\sqrt{3}}$ d. $\frac{1}{\sqrt{6}}$

Q 22. **The distance between lines $\vec{r}=\vec{a}_1+t\vec{b}$ and $\vec{r}=\vec{a}_2+s\vec{b}$ is:**

a. $|(\vec{a}_2-\vec{a}_1)\times\vec{b}|$ b. $\frac{|(\vec{a}_2-\vec{a}_1)\times\vec{b}|}{|\vec{b}|}$

c. $\frac{|(\vec{a}_2-\vec{a}_1)\times\vec{b}|}{|\vec{a}_2-\vec{a}_1|}$ d. $\frac{|(\vec{a}_2-\vec{a}_1)\times\vec{b}|}{|\vec{a}_2-\vec{a}_1|\cdot|\vec{b}|}$

Assertion & Reason Type Questions

Directions (Q. Nos. 23-29): *In the following questions, each question contains Assertion (A) and Reason (R). Each question has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are:*

a. Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation for Assertion (A)

b. Both Assertion (A) and Reason (R) are true but Reason (R) is not the correct explanation of Assertion (A)

c. Assertion (A) is true but Reason (R) is false

d. Assertion (A) is false but Reason (R) is true

Q 23. **Assertion (A): The points (1, 2, 3), (−2, 3, 4) and (7, 0, 1) are collinear.**

Reason (R): If a line makes angles $\frac{\pi}{2}, \frac{3\pi}{4}$ and $\frac{\pi}{4}$ with X, Y and Z-axes respectively, then its direction cosines are $0, \frac{-1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}$.

Q 24. **Assertion (A): If the cartesian equation of a line is $\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-6}{2}$, then its vector form is $\vec{r}=5\hat{i}-4\hat{j}+6\hat{k}+\lambda(3\hat{i}+7\hat{j}+2\hat{k})$.**

Reason (R): The cartesian equation of the line which passes through the point (−2, 4, −5) and parallel to the line given by $\frac{x+3}{3}=\frac{y-4}{5}=\frac{z+8}{6}$, is $\frac{x+3}{-2}=\frac{y-4}{4}=\frac{z+8}{-5}$.

Q 25. **Assertion (A): The acute angle between the line $\vec{r}=\hat{i}+\hat{j}+2\hat{k}+\lambda(\hat{i}-\hat{j})$ and the X-axis is $\frac{\pi}{4}$.**

Reason (R): The acute angle θ between the lines $\vec{r}=x_1\hat{i}+y_1\hat{j}+z_1\hat{k}+\lambda(a_1\hat{i}+b_1\hat{j}+c_1\hat{k})$ and $\vec{r}=x_2\hat{i}+y_2\hat{j}+z_2\hat{k}+\mu(a_2\hat{i}+b_2\hat{j}+c_2\hat{k})$ is given by $\cos\theta=\frac{|a_1a_2+b_1b_2+c_1c_2|}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$.

(CBSE SQP 2022-23)

Q 26. **Assertion (A): The three lines with direction cosines $\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}; \frac{4}{13}, \frac{12}{13}, \frac{3}{13}; \frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$ are mutually perpendicular.**

Reason (R): The line through the points (1, −1, 2) and (3, 4, −2) is perpendicular to the line through the points (0, 3, 2) and (3, 5, 6).

Q 27. **Assertion (A): The pair of lines given by**

$\vec{r}=\hat{i}-\hat{j}+\lambda(2\hat{i}+\hat{k})$

and $\vec{r}=2\hat{i}-\hat{k}+\mu(\hat{i}+\hat{j}-\hat{k})$ **intersect.**

Reason (R): Two lines intersect each other, if they are not parallel and shortest distance = 0.

Q 28. Consider the lines

$L_1: \frac{x+1}{3} = \frac{y+2}{-1} = \frac{z+1}{2}, L_2: \frac{x-2}{-1} = \frac{y+2}{3} = \frac{z-3}{3}$

Assertion (A): The lines L_1 and L_2 are mutually perpendicular.

Reason (R): The unit vector perpendicular to both the lines L_1 and L_2 is $\frac{-9\hat{i}-11\hat{j}+8\hat{k}}{\sqrt{266}}$.

Q 29. Assertion (A): The lines $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu\vec{b_2}$ are perpendicular, when $\vec{b_1} \cdot \vec{b_2} = 0$.

Reason (R): The angle θ between the lines $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu\vec{b_2}$ is given by $\cos\theta = \frac{\vec{b_1} \cdot \vec{b_2}}{|\vec{b_1}||\vec{b_2}|}$.

(CBSE 2023)

Answers

1. (a)	2. (a)	3. (d)	4. (d)	5. (b)	6. (b)	7. (d)	8. (b)	9. (b)	10. (c)
11. (a)	12. (a)	13. (d)	14. (b)	15. (d)	16. (a)	17. (b)	18. (d)	19. (a)	20. (d)
21. (d)	22. (b)	23. (b)	24. (c)	25. (a)	26. (b)	27. (a)	28. (b)	29. (a)	

Case Study Based Questions

Case Study 1

Two motorcycles X and Y are running at the speed more than allowed speed on the road along the lines $\vec{r} = \lambda(\hat{i}+2\hat{j}-\hat{k})$ and $\vec{r} = 3\hat{i}+3\hat{j}+\mu(2\hat{i}+\hat{j}+\hat{k})$ respectively.

Based on the above information, solve the following questions:

Q 1. The cartesian equation of the line along which motorcycle X is running, is:

a. $\frac{x+1}{1} = \frac{y+1}{2} = \frac{z-1}{-1}$ b. $\frac{x}{1} = \frac{y}{2} = \frac{z}{-1}$

c. $\frac{x}{1} = \frac{y}{2} = \frac{z}{1}$ d. None of these

Q 2. The direction cosines of line along which motorcycle X is running, are:

a. $<1, -2, 1>$ b. $<1, 2, -1>$

c. $<\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}>$ d. $<\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}>$

Q 3. The direction ratios of the line along which motorcycle Y is running, are:

a. $<1, 0, 2>$ b. $<2, 1, 0>$

c. $<1, 1, 2>$ d. $<2, 1, 1>$

Q 4. The shortest distance between the given lines is:

a. 4 units   b. $2\sqrt{3}$ units

c. $3\sqrt{2}$ units   d. zero

Q 5. The motorcycles will meet with an accident at the point:

a. $(-1, 1, 2)$ b. $(2, 1, -1)$

c. $(1, 2, -1)$ d. None of these

Solutions

1. The line along which motorcycle X is running, is $\vec{r} = \lambda(\hat{i}+2\hat{j}-\hat{k})$, which can rewrite as:

$$(x\hat{i}+y\hat{j}+z\hat{k}) = \lambda\hat{i}+2\lambda\hat{j}-\lambda\hat{k}$$

$$\Rightarrow \quad x = \lambda,\ y = 2\lambda,\ z = -\lambda$$

$$\Rightarrow \quad \frac{x}{1} = \frac{y}{2} = \frac{z}{-1} = \lambda$$

Thus, the required cartesian equation is $\frac{x}{1} = \frac{y}{2} = \frac{z}{-1}$.

So, option (b) is correct.

2. Clearly, direction ratios of the required line are $<1, 2, -1>$.

$\therefore$ Direction cosines are:

$$<\frac{1}{\sqrt{(1)^2+(2)^2+(-1)^2}}, \frac{2}{\sqrt{(1)^2+(2)^2+(-1)^2}}, \frac{-1}{\sqrt{(1)^2+(2)^2+(-1)^2}}>$$

i.e., $<\frac{1}{\sqrt{1+4+1}}, \frac{2}{\sqrt{1+4+1}}, \frac{-1}{\sqrt{1+4+1}}>$

i.e., $<\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}>$

So, option (d) is correct.

3. The line along which motorcycle B is running, is $\vec{r} = (3\hat{i}+3\hat{j})+\mu(2\hat{i}+\hat{j}+\hat{k})$, which is parallel to the vector $2\hat{i}+\hat{j}+\hat{k}$.

$\therefore$ D.R.'s of the required line are $<2, 1, 1>$.

So, option (d) is correct.

4. Here, $\vec{a_1} = 0\hat{i} + 0\hat{j} + 0\hat{k}, \vec{a_2} = 3\hat{i} + 3\hat{j}, \vec{b_1} = \hat{i} + 2\hat{j} - \hat{k}$

and $\vec{b_2} = 2\hat{i} + \hat{j} + \hat{k}$

$\therefore \quad \vec{a_2} - \vec{a_1} = (3\hat{i} + 3\hat{j}) - (0\hat{i} + 0\hat{j} + 0\hat{k}) = 3\hat{i} + 3\hat{j}$

and $\quad \vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ 2 & 1 & 1 \end{vmatrix}$

$$= (2+1)\hat{i} - (1+2)\hat{j} + (1-4)\hat{k}$$
$$= 3\hat{i} - 3\hat{j} - 3\hat{k}$$

Now, $(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2}) = (3\hat{i} + 3\hat{j}) \cdot (3\hat{i} - 3\hat{j} - 3\hat{k})$

$$= (3)(3) + (3)(-3) + (0)(-3) = 9 - 9 = 0$$

Hence, shortest distance between the given lines is 0.

So, option (d) is correct.

5. Cartesian equation of Motorcycle x is

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{-1} \quad \text{...(1)}$$

and cartesian equation of Motorcycle y is

$$\frac{x-3}{2} = \frac{y-3}{1} = \frac{z}{1} \quad \text{...(2)}$$

Take a point $(x, y, z) = (1, 2, -1)$

from eq. (1), $\frac{1}{1} = \frac{2}{2} = \frac{-1}{-1} \Rightarrow 1 = 1 = 1$ (true)

from eq. (2), $\frac{1-3}{2} = \frac{2-3}{1} = \frac{-1}{1} \Rightarrow \frac{-2}{2} = \frac{-1}{1} = \frac{-1}{1}$

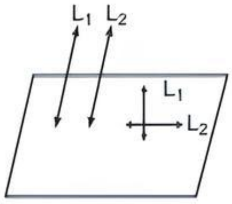
$\Rightarrow 1 = 1 = 1$ (true)

Since, the point $(1, 2, -1)$ satisfy both the equations of lines, therefore point of intersection of given lines is $(1, 2, -1)$.

So, the Motorcycles will meet with an accident at the point $(1, 2, -1)$.

So, option (c) is correct.

Case Study 2



If a_1, b_1, c_1 and a_2, b_2, c_2 are direction ratios of two lines say L_1 and L_2 respectively.

Then $L_1 || L_2$ iff $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

and $L_1 \perp L_2$ iff $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

Based on the above information, solve the following questions:

Q 1. If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of L_1 and L_2 respectively, then L_1 will be perpendicular to L_2, iff:

a. $l_1l_2 + m_1m_2 + n_1n_2 = 0$

b. $l_1m_2 + m_1l_2 + n_1n_2 = 0$

c. $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$

d. None of the above

Q 2. If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of L_1 and L_2 respectively, then L_1 will be parallel to L_2 iff:

a. $l_1l_2 + m_1m_2 + n_1n_2 = 0$

b. $l_1m_2 + m_1l_2 + n_1n_2 = 0$

c. $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$

d. None of the above

Q 3. The coordinates of the foot of the perpendicular drawn from the point A (2, 1, 2) to the line $\frac{x-3}{1} = \frac{y-2}{1} = \frac{z-5}{1}$ are:

a. $\left(\frac{4}{3}, \frac{1}{3}, \frac{10}{3}\right)$ b. (2, 4, 5)

c. (3, 4, 5) d. (4, 3, 5)

Q 4. The direction ratios of the line which is perpendicular to the lines with direction ratios proportional to (–1, 3, 2) and (4, 0, –3), are:

a. $< -9, 5, -12 >$ b. $< 1, 2, 1 >$

c. $< 2, -1, 2 >$ d. $< -1, 2, 2 >$

Q 5. The lines $\frac{-x+2}{-3} = \frac{-y+1}{2} = \frac{z-2}{0}$ and $\frac{-x+1}{-1} = \frac{2y+3}{3} = \frac{z+5}{2}$ are:

a. parallel b. perpendicular

c. skew-lines d. non-intersecting

Solutions

1. Since, D.R.'s are proportional to D.C.'s, therefore L_1 will be perpendicular to L_2 iff

$$l_1l_2 + m_1m_2 + n_1n_2 = 0$$

So, option (a) is correct.

2. Since, D.R.'s are proportional to D.C.'s, therefore L_1 will be parallel to L_2, iff

$$\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$$

So, option (c) is correct.

3. Equation of line is:

$$\frac{x-3}{1} = \frac{y-2}{1} = \frac{z-5}{1}$$

Let coordinates of foot of perpendicular be D (x, y, z).

$\therefore$ Direction ratios of AD are $< x-2, y-1, z-2 >$

Also AD is perpendicular to the given line

$\therefore \quad 1(x-2) + 1(y-1) + 1(z-2) = 0$

$\Rightarrow \quad x - 2 + y - 1 + z - 2 = 0$

$\Rightarrow \quad x + y + z = 5 \quad \text{...(1)}$

Any point on the line is $(\lambda + 3, \lambda + 2, \lambda + 5)$.

This will satisfy the eq. (1).

$\lambda + 3 + \lambda + 2 + \lambda + 5 = 5 \Rightarrow 3\lambda = 5 - 10$

$\Rightarrow \quad \lambda = -\frac{5}{3}$

$\therefore$ Required coordinate of foot of perpendicular is

$\left(-\frac{5}{3}+3, -\frac{5}{3}+2, -\frac{5}{3}+5\right)$

i.e., $\left(\frac{4}{3}, \frac{1}{3}, \frac{10}{3}\right)$

So, option (a) is correct.

4. Let a, b, c be the direction ratios of the required line. Since it is perpendicular to the lines whose direction ratios are $(-1, 3, 2)$ and $(4, 0, -3)$ respectively.

$\therefore \quad -a+3b+2c=0 \quad \ldots(1)$

and $\quad 4a+0\cdot b-3c=0 \quad \ldots(2)$

On solving eqs. (1) and (2) by cross-multiplication, we get

$$\frac{a}{-9+0}=\frac{b}{8-3}=\frac{c}{0-12} \Rightarrow \frac{a}{-9}=\frac{b}{5}=\frac{c}{-12}$$

Thus, the direction ratios of the required line are

$$<-9, 5, -12>.$$

So, option (a) is correct.

5. Given lines are

$$\frac{-x+2}{-3}=\frac{-y+1}{2}=\frac{z-2}{0}$$

or $\quad \frac{x-2}{3}=\frac{y-1}{-2}=\frac{z-2}{0}$

and $\quad \frac{-x+1}{-1}=\frac{2y+3}{3}=\frac{z+5}{2}$

or $\quad \frac{x-1}{1}=\frac{y-(-3/2)}{3/2}=\frac{z-(-5)}{2}$

$\therefore$ Direction ratios of given lines are $<3, -2, 0>$ and $<1, 3/2, 2>$.

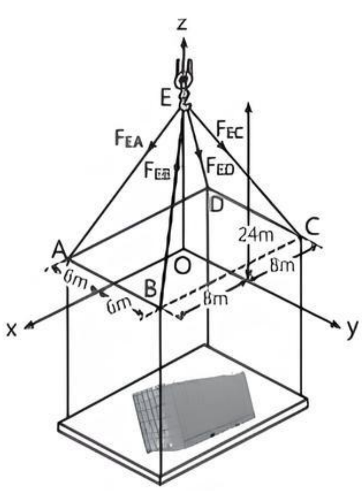
Now, as $(3)(1)+(-2)(3/2)+(0)(2)=3-3+0=0$

Thus, given lines are perpendicular to each other.

So, option (b) is correct.

Case Study 3

Consider the following diagram, where the forces in the cable are given:



Based on the above information, solve the following questions:

Q 1. The vector $\overrightarrow{ED}$ is:

a. $8\hat{i}-6\hat{j}+24\hat{k}$ b. $-8\hat{i}-6\hat{j}+24\hat{k}$

c. $-8\hat{i}-6\hat{j}-24\hat{k}$ d. $8\hat{i}+6\hat{j}+24\hat{k}$

Q 2. The length of the cable EB is:

a. 24 units b. 26 units

c. 27 units d. 25 units

Q 3. The length of cable EC is equal to the length of:

a. EA b. EB

c. ED d. All of these

Q 4. The sum of all vectors along the cables is:

a. $96\hat{i}$ b. $96\hat{j}$

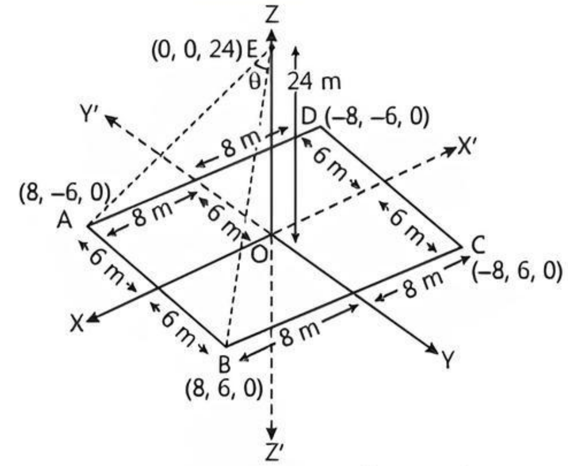
c. $-96\hat{k}$ d. $96\hat{k}$

Q 5. The angle between $\overrightarrow{EA}$ and $\overrightarrow{EB}$ is:

a. $\frac{\pi}{3}$ b. $\frac{\pi}{6}$

c. $\cos^{-1}\left(\frac{57}{71}\right)$ d. $\cos^{-1}\left(\frac{151}{169}\right)$

Solutions



1. Clearly, the coordinates of D are $(-8, -6, 0)$ and that of E are $(0, 0, 24)$.

$\therefore$ Vector $\overrightarrow{ED}$ is $(-8-0)\hat{i}+(-6-0)\hat{j}+(0-24)\hat{k}$,

i.e., $-8\hat{i}-6\hat{j}-24\hat{k}$.

So, option (c) is correct.

2. Since, the coordinates of B are $(8, 6, 0)$ and that of E are $(0, 0, 24)$, therefore length of cable

$$EB=\sqrt{(8-0)^2+(6-0)^2+(0-24)^2}$$
$$=\sqrt{64+36+576}=\sqrt{676}=26 \text{ units.}$$

So, option (b) is correct.

3. Since, the coordinates of C are $(-8, 6, 0)$, therefore length of cable $EC=\sqrt{(-8-0)^2+(6-0)^2+(0-24)^2}$

$$=\sqrt{64+36+576}=\sqrt{676}=26 \text{ units.}$$

Similarly, length of cable EA = ED = EB = 26 units.

So, option (d) is correct.

4. Sum of all vectors along the cables

$$=\overrightarrow{EA}+\overrightarrow{EB}+\overrightarrow{EC}+\overrightarrow{ED}$$
$$=(8\hat{i}-6\hat{j}-24\hat{k})+(8\hat{i}+6\hat{j}-24\hat{k})$$
$$+(-8\hat{i}+6\hat{j}-24\hat{k})+(-8\hat{i}-6\hat{j}-24\hat{k})$$
$$=-96\hat{k}$$

So, option (c) is correct.

5. Let the angle between $\vec{EA}$ and $\vec{EB}$ be θ

$\vec{EA} = 8\hat{i} - 6\hat{j} - 24\hat{k}$ and $\vec{EB} = 8\hat{i} + 6\hat{j} - 24\hat{k}$

$$\therefore \cos\theta = \frac{|\vec{EA}\cdot\vec{EB}|}{|\vec{EA}||\vec{EB}|} = \frac{|(8\hat{i}-6\hat{j}-24\hat{k})\cdot(8\hat{i}+6\hat{j}-24\hat{k})|}{|8\hat{i}-6\hat{j}-24\hat{k}||8\hat{i}+6\hat{j}-24\hat{k}|}$$

$$= \frac{|(8)(8)+(-6)(6)+(-24)(-24)|}{\sqrt{(8)^2+(-6)^2+(-24)^2}\sqrt{(8)^2+(6)^2+(-24)^2}}$$

$$= \frac{|64-36+576|}{\sqrt{64+36+576}\sqrt{64+36+576}} = \frac{604}{\sqrt{676}\sqrt{676}}$$

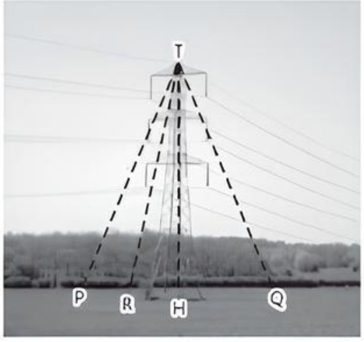
$$= \frac{604}{676} = \frac{151}{169}$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{151}{169}\right)$$

So, option (d) is correct.

Case Study 4

An electricity tower stands on an agricultural horizontal field, where agricultural activities are not obstructed in any manner. Consider the surface on which the electricity tower stands as a plane having points P (2,–1,3), Q (0,4,1) R (2,1,–1) and H (0, –1, 2) on it. The electricity tower is tied with three cables from the points P, Q and R such that it stands vertically on the field. The top of the electricity tower is at the point T (4, 1, 3) as shown in the following figure:



Based on the above information, solve the following questions:

Q 1. Find the equation of the perpendicular line drawn from the top of the electricity tower to the horizontal field.

Q 2. Find the distance between the points P and Q.

Q 3. If the points P, Q and R connect each other with a wire and form a triangle, then find the area of ΔPQR.

Or

Check the points P, Q and R collinear or not.

Solutions

1. The equation of the perpendicular line TH drawn from the top T (4, 1, 3) of the electricity tower to the horizontal field at point H is

$$\frac{x-4}{4} = \frac{y-1}{2} = \frac{z-3}{1}$$

where, < 4, 2, 1 > are direction ratios of the line TH.

2. The distance between the points, P and Q is

$$PQ = \sqrt{(0-2)^2+(4+1)^2+(1-3)^2}$$

$$= \sqrt{4+25+4} = \sqrt{33}$$

3. Since, P (2, –1, 3), Q (0, 4, 1) and R (2, 1, –1)

$$\Delta_x = \frac{1}{2}\begin{vmatrix} -1 & 3 & 1 \\ 4 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix}$$

$$= \frac{1}{2}[-1(1+1)-3(4-1)+1(-4-1)]$$

$$= \frac{1}{2}[-2-9-5]$$

$$= \frac{-16}{2} = -8$$

$$\Delta_y = \frac{1}{2}\begin{vmatrix} 2 & 3 & 1 \\ 0 & 1 & 1 \\ 2 & -1 & 1 \end{vmatrix}$$

$$= \frac{1}{2}[2(1+1)-3(0-2)+1(0-2)]$$

$$= \frac{1}{2}[4+6-2] = \frac{8}{2} = 4$$

and

$$\Delta_z = \frac{1}{2}\begin{vmatrix} 2 & -1 & 1 \\ 0 & 4 & 1 \\ 2 & 1 & 1 \end{vmatrix}$$

$$= \frac{1}{2}[2(4-1)+1(0-2)+1(0-8)]$$

$$= \frac{1}{2}[6-2-8] = -\frac{4}{2} = -2$$

$$\therefore \text{ Area of } \Delta PQR = \sqrt{\Delta_x^2 + \Delta_y^2 + \Delta_z^2}$$

$$= \sqrt{(-8)^2+(4)^2+(-2)^2}$$

$$= \sqrt{64+16+4} = \sqrt{84}$$

$$= 2\sqrt{21} \text{ sq. units.}$$

Or

Given, points are

$P(2, -1, 3) = (x_1, y_1, z_1)$,

$Q(0, 4, 1) = (x_2, y_2, z_2)$

and $R(2, 1, -1) = (x_3, y_3, z_3)$

If the points are collinear then

$$\frac{x_3-x_1}{x_2-x_1} = \frac{y_3-y_1}{y_2-y_1} = \frac{z_3-z_1}{z_2-z_1}$$

$$\Rightarrow \quad \frac{2-2}{0-2} = \frac{1+1}{4+1} = \frac{-1-3}{1-3}$$

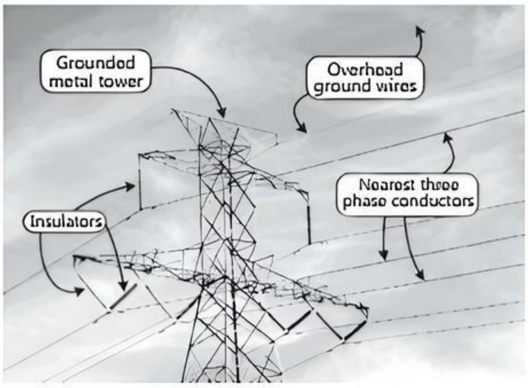
$$\Rightarrow \quad \frac{0}{-2} = \frac{2}{5} = \frac{-4}{-2}$$

$$\Rightarrow \quad 0 \neq \frac{2}{5} \neq 2$$

Thus, the points P, Q and R are not collinear.

Case Study 5

Electrical transmission wires which are laid down in winters are stretched tightly to accommodate expansion in summers.



Two such wires lie along the following lines:

$$l_1 : \frac{x+1}{3} = \frac{y-3}{-2} = \frac{z+2}{-1}$$

$$l_2 : \frac{x}{-1} = \frac{y-7}{3} = \frac{z+7}{-2}$$

Based on the given information, solve the following questions:

Q 1. Find the angle between the lines l_1 and l_2.

Q 2. Find the point of intersection of the lines l_1 and l_2.

Solutions

1. The direction ratios of given lines are $3, -2, -1$ and $-1, 3, -2$ respectively.

TR!CK

The angle between two lines

$$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$$

and $$\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$$ *is given by.*

$$\cos\theta = \left| \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}} \right|$$

$$\therefore \cos\theta = \left| \frac{(3)(-1)+(-2)(3)+(-1)(-2)}{\sqrt{(3)^2+(-2)^2+(-1)^2}\sqrt{(-1)^2+(3)^2+(-2)^2}} \right|$$

$$= \left| \frac{-3-6+2}{\sqrt{9+4+1}\sqrt{1+9+4}} \right| = \left| \frac{-7}{\sqrt{14}\sqrt{14}} \right|$$

$$= \frac{7}{14} = \frac{1}{2} = \cos 60^\circ$$

$$\Rightarrow \quad \theta = 60^\circ = \frac{\pi}{3}$$

2. From line (1),

$$\frac{x+1}{3} = \frac{y-3}{-2} = \frac{z+2}{-1} = \lambda \quad \text{(say)}$$

$\Rightarrow x = 3\lambda - 1,\ y = -2\lambda + 3$ and $z = -\lambda - 2$.

So, the coordinates of a general point on this line are $(3\lambda-1, -2\lambda+3, -\lambda-2)$.

From line (2),

$$\frac{x}{-1} = \frac{y-7}{3} = \frac{z+7}{-2} = \mu \quad \text{(say)}$$

So, the coordinates of a general point on this line are $(-\mu, 3\mu+7, -2\mu-7)$.

If the line intersects, then they have a common point.

So, for some values of λ and μ, we must have

$$3\lambda - 1 = -\mu,\ -2\lambda + 3 = 3\mu + 7,\ -\lambda - 2 = -2\mu - 7$$

$$\Rightarrow \quad 3\lambda + \mu = 1, 3\mu + 2\lambda = -4, \lambda - 2\mu = 5$$

On solving first two equations, we get $\lambda = 1$ and $\mu = -2$

Since, $\lambda = 1$ and $\mu = -2$ satisfy the third equation. So the given lines intersect. On putting $\lambda = 1$ in $(3\lambda-1, -2\lambda+3, -\lambda-2)$, the coordinates of the required point of intersection are $(2, 1, -3)$.

Case Study 6

The Indian Coast Guard (ICG) while patrolling, saw a suspicious boat with four men. They were no way looking like fishermen. The soldiers were closely observing the movement of the boat for an opportunity to seize the boat. They observe that the boat is moving along a planar surface. At an instant of time, the coordinates of the position of coast guard helicopter and boat are A (4, 5, 2) and B (1, 2, 3) respectively.

Based on the above information, solve the following questions:

Q 1. If the soldier decides to shoot the boat at given instant of time, where the distance measured in metres, then what is the distance that bullet has to travel?

Q 2. If the speed of bullet is 45 m/s, then how much time will the bullet take to hit the boat after the shot is fired?

Q 3. Find the direction cosines of line passing through the positions of helicopter and boat.

Or

At the given instant of time, find the equation of line passing through the positions of helicopter and boat.

Solutions

1. Required distance = Distance between A and B

$$= \sqrt{(1-4)^2 + (2-5)^2 + (3-2)^2}$$

$$= \sqrt{9+9+1} = \sqrt{19} \text{ m}$$

2. We know that,

$$\text{Distance} = \text{Speed} \times \text{Time}$$

$\therefore$ Required time $= \dfrac{\text{Distance}}{\text{Speed}} = \dfrac{\sqrt{19}}{45}$ sec

3. Direction ratios of line AB are $< 1-4, 2-5, 3-2 >$; i.e; $< -3, -3, 1 >$ From part (1), length of AB is $\sqrt{19}$ m.

$\therefore$ Direction cosines of line AB are $< \dfrac{-3}{\sqrt{19}}, \dfrac{-3}{\sqrt{19}}, \dfrac{1}{\sqrt{19}} >$

Or

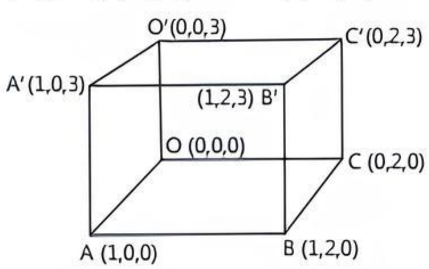
$\therefore$ Required equation of line *AB* is

$$\frac{x-4}{-3} = \frac{y-5}{-3} = \frac{z-2}{1}$$

[$\because$ Dr's of line AB are $< -3, -3, 1 >$]

Case Study 7

In a diamond exhibition, a diamond is covered in cuboidal glass box having coordinates O (0, 0, 0), A (1, 0, 0), B (1, 2, 0), C (0, 2, 0), O′ (0, 0, 3), A′ (1, 0, 3), B′ (1, 2, 3) and C′ (0, 2, 3).



Based on the above information, solve the following questions:

Q 1. Find the direction ratios of OA′.

Q 2. Find the equation of diagonal OB′.

Or

Find the length of the longest rod place in cuboidal glass box.

Q 3. Find the angle between OB and OB′.

Solutions

1. Direction ratios of OA′ are

$< 1-0, 0-0, 3-0 >$ *i.e.*, $< 1, 0, 3 >$

2. Since Dr's of OB′ are $< 1-0, 2-0, 3-0 >$ i.e., $< 1, 2, 3 >$

$\therefore$ Equation of diagonal OB′ is

$$\frac{x-0}{1} = \frac{y-0}{2} = \frac{z-0}{3} \text{ i.e., } \frac{x}{1} = \frac{y}{2} = \frac{z}{3}$$

Or

Length of the longest rod

$= \sqrt{(1)^2 + (2)^2 + (3)^2} = \sqrt{14}$ units

3. Direction ratios of OB are

$< a_1, b_1, c_1 > = < 1-0, 2-0, 0-0 > = < 1, 2, 0 >$

and direction ratios of OB′ are

$< a_2, b_2, c_2 > = < 1-0, 2-0, 3-0 > = < 1, 2, 3 >$.

Let θ be the angle between OB and OB′, then

$$\cos\theta = \frac{|a_1 a_2 + b_1 b_2 + c_1 c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$\Rightarrow \quad \cos\theta = \frac{|(1)(1) + (2)(2) + (0)(3)|}{\sqrt{1^2 + 2^2 + 0^2}\sqrt{1^2 + 2^2 + 3^2}} = \frac{1+4+0}{\sqrt{5}\sqrt{14}}$$

$$\Rightarrow \quad \cos\theta = \frac{5}{\sqrt{5}\sqrt{14}} = \sqrt{\frac{5}{14}}$$

$$\therefore \quad \theta = \cos^{-1}\left(\sqrt{\frac{5}{14}}\right)$$

Very Short Answer Type Questions

Q 1. Find the direction cosines of *X*, *Y* and *Z*-axes. *(NCERT EXERCISE)*

Q 2. Write the distance of the point (3, −5, 12) from *X*-axis. *(CBSE 2017)*

Q 3. Find the direction cosines of the line passing through the two points (−2, 4, −5) and (1, 2, 3). *(NCERT EXERCISE)*

Q 4. Find the vector equation of the line which passes through the point (3, 4, 5) and is parallel to the vector $2\hat{i} + 2\hat{j} - 3\hat{k}$. *(CBSE 2019)*

Q 5. Find the cartesian equation of a line which passes through the point (−2, 4, −5) and parallel to the line $\dfrac{x+3}{3} = \dfrac{y-4}{5} = \dfrac{z-8}{6}$. *(NCERT EXERCISE)*

Q 6. Find the angle between the lines

$$\frac{x-4}{1} = \frac{y+5}{2} = \frac{z-3}{1} \text{ and } \frac{x}{2} = \frac{y}{3} = \frac{z}{1}.$$

Q 7. Show that the lines $\dfrac{x-5}{7} = \dfrac{y+2}{-5} = \dfrac{z}{1}$ and $\dfrac{x}{1} = \dfrac{y}{2} = \dfrac{z}{3}$ are perpendicular. *(NCERT EXERCISE)*

Short Answer Type-I Questions

Q 1. Prove that the points A (1, 2, 7), B (2, 6, 3) and C (3, 10, −1) are collinear.

Q 2. The *x*-coordinate of a point on the line joining the points P (2, 2, 1) and Q (5, 1, −2) is 4. Find its *z*-coordinate. *(CBSE 2017)*

Q 3. The equations of a line are $5x - 3 = 15y + 7 = 3 - 10z$. Write the direction cosines of the line and find the coordinates of a point through which it passes. *(CBSE 2023)*

Q 4. Find the direction ratio and direction cosines of a line parallel to the line whose equations are $6x - 12 = 3y + 9 = 2z - 2$. *(CBSE SQP 2022-23)*

Q 5. Find the vector and the cartesian equations of a line that passes through the point *A*(1, 2, −1) and parallel to the line $5x - 25 = 14 - 7y = 35z$. *(CBSE 2023)*

Or

Find the vector equation of the line passing through the point A (1, 2, –1) and parallel to the line $5x-25=14-7y=35z$. *(CBSE 2017)*

Q 6. Find the angle between the following two lines:

$\vec{r}=2\hat{i}-5\hat{j}+\hat{k}+\lambda(3\hat{i}+2\hat{j}+6\hat{k})$;

$\vec{r}=7\hat{i}-6\hat{k}+\mu(\hat{i}+2\hat{j}+2\hat{k})$ *(CBSE 2023)*

Q 7. Using vectors, find the angle between the lines $\frac{x+3}{3}=\frac{y-1}{5}=\frac{z+3}{4}$ and $\frac{x+1}{1}=\frac{y-4}{1}=\frac{z-5}{2}$.

Q 8. Find the value of p if lines $\frac{1-x}{3}=\frac{7y-14}{2p}=\frac{z-3}{2}$ and $\frac{7-7x}{3p}=\frac{y-5}{1}=\frac{6-z}{5}$ are perpendicular to each other. *(NCERT EXERCISE)*

Q 9. Find the vector equation of the line passing through the point (2, 1, 3) and perpendicular to both the lines *(CBSE 2023)*

$\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{3}$; $\frac{x}{-3}=\frac{y}{2}=\frac{z}{5}$.

Short Answer Type-II Questions

Q 1. Find the value of λ, so that the lines $\frac{1-x}{3}=\frac{7y-14}{\lambda}=\frac{z-3}{2}$ and $\frac{7-7x}{3\lambda}=\frac{y-5}{1}=\frac{6-z}{5}$ are at right angles. Also, find whether the lines are intersecting or not. *(CBSE 2019)*

Q 2. Show that the lines $\frac{x+1}{3}=\frac{y+3}{5}=\frac{z+5}{7}$ and $\frac{x-2}{1}=\frac{y-4}{3}=\frac{z-6}{5}$ intersect. Also, find the point of intersection of lines.

Q 3. Find the co-ordinates of the point on the line $\frac{x+2}{3}=\frac{y+1}{2}=\frac{z-3}{2}$ which is at a distance of 5 units from the point (1, 3, 3). *(CBSE 2022 Term-2)*

Q 4. Find the coordinates of the foot of the perpendicular drawn from point (5, 7, 3) to the line $\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}$. *(CBSE 2023)*

Q 5. Find the length of foot of perpendicular drawn from point (1, 2, 3) to the line $\frac{x-6}{3}=\frac{y-7}{2}=\frac{z-7}{-2}$. Also, find the equation of perpendicular. *(NCERT EXEMPLAR)*

Q 6. Find the shortest distance between the lines $\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}$ and $\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}$.

(NCERT EXEMPLAR, NCERT EXERCISE)

Q 7. Find the shortest distance between the lines $\vec{r}=(4\hat{i}-\hat{j})+\lambda(\hat{i}+2\hat{j}-3\hat{k})$ and $\vec{r}=(\hat{i}-\hat{j}+2\hat{k})+\mu(2\hat{i}+4\hat{j}-5\hat{k})$. *(CBSE 2018)*

Q 8. Find the shortest distance between the following lines:

$\vec{r}=3\hat{i}+5\hat{j}+7\hat{k}+\lambda(\hat{i}-2\hat{j}+\hat{k})$ and

$\vec{r}=(-\hat{i}-\hat{j}-\hat{k})+\mu(7\hat{i}-6\hat{j}+\hat{k})$. *(CBSE 2022 Term-2)*

Q 9. Find the distance between the lines: *(CBSE 2023)*

$\vec{r}=(\hat{i}+2\hat{j}-4\hat{k})+\lambda(2\hat{i}+3\hat{j}+6\hat{k})$;

$\vec{r}=(3\hat{i}+3\hat{j}-5\hat{k})+\mu(4\hat{i}+6\hat{j}+12\hat{k})$

Q 10. Find the shortest distance between the lines $\frac{x}{m_1}=\frac{y}{1}=\frac{z-a}{0}$ and $\frac{x}{m_2}=\frac{y}{1}=\frac{z+a}{0}$.

Q 11. Find the equation of line which passes through the origin and intersect the two lines $\frac{x-1}{2}=\frac{y+3}{4}=\frac{z-5}{3}$ and $\frac{x-4}{2}=\frac{y+3}{3}=\frac{z-14}{4}$.

Q 12. Show that the lines

$\vec{r}=(\hat{i}+2\hat{j}+3\hat{k})+\lambda_1(2\hat{i}+3\hat{j}+4\hat{k})$ and

$\vec{r}=(2\hat{i}+3\hat{j}+4\hat{k})+\lambda_2(3\hat{i}+4\hat{j}+5\hat{k})$ intersect each other. Also, find the point of intersection.

Long Answer Type Questions

Q 1. Show that the following lines do not intersect each other:

$\frac{x-1}{3}=\frac{y+1}{2}=\frac{z-1}{5}$; $\frac{x+2}{4}=\frac{y-1}{3}=\frac{z+1}{-2}$

(CBSE 2023)

Q 2. Find the image of the point (2, –1, 5) in the line $\frac{x-11}{10}=\frac{y+2}{-4}=\frac{z+8}{-11}$. *(CBSE 2023)*

Q 3. Vertices B and C of ΔABC lie on the line $\frac{x+2}{2}=\frac{y-1}{1}=\frac{z}{4}$. Find the area of ΔABC given that point A has coordinates (1, –1, 2) and the line segment BC has length of 5 units. *(CBSE 2023)*

Q 4. Find the vector and cartesian equations of the line which is perpendicular to the lines with equations $\frac{x+2}{1}=\frac{y-3}{2}=\frac{z+1}{4}$ and $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and passes through the point (1, 1, 1). Also, find the angle between the given lines. *(CBSE 2020)*

Q 5. The equations of motion of a rocket are: $x = 2t, y = -4t, z = 4t$, where the time t is given in seconds and the coordinates of a moving point in km. What is the path of the rocket? At what distances will the rocket be from the starting point O (0, 0, 0) and from the following line in 10 seconds?

$\vec{r} = 20\hat{i} - 10\hat{j} + 40\hat{k} + \mu(10\hat{i} - 20\hat{j} + 10\hat{k})$

(CBSE SQP 2022-23)

Q 6. An insect is crawling along the line $\vec{r} = 6\hat{i} + 2\hat{j} + 2\hat{k} + \lambda(\hat{i} - 2\hat{j} + 2\hat{k})$ and another insect is crawling along the line $\vec{r} = -4\hat{i} - \hat{k} + \mu(3\hat{i} - 2\hat{j} - 2\hat{k})$. At what points on the lines should they reach so that the distance between them is the shortest? Find the shortest possible distance between them. (CBSE SQP 2022-23)

Q 7. Find the coordinates of the image of the point (1, 6, 3) with respect to the line $\vec{r} = (\hat{j} + 2\hat{k}) + \lambda(\hat{i} + 2\hat{j} + 3\hat{k})$; where '$\lambda$' is a scalar. Also, find the distance of the image from the Y-axis. (CBSE SQP 2023-24)

Q 8. An aeroplane is flying along the line $\vec{r} = \lambda(\hat{i} - \hat{j} + \hat{k})$; where '$\lambda$' is a scalar and another aeroplane is flying along the line $\vec{r} = \hat{i} - \hat{j} + \mu(-2\hat{j} + \hat{k})$; where '$\mu$' is a scalar. At what points on the lines should they reach, so that the distance between there is the shortest? Find the shortest possible distance between them.

(CBSE SQP 2023-24)

Solutions

Very Short Answer Type Questions

1. X-axis makes the angles with X, Y and Z-axes are 0°, 90° and 90° respectively.

So, direction cosines of X-axis are

$\cos 0°, \cos 90°, \cos 90°$ *i.e.*, 1, 0, 0.

Similarly, direction cosines of Y-axis are

$\cos 90°, \cos 0°, \cos 90°$ *i.e.*, 0, 1, 0.

and direction cosines of Z-axis are

$\cos 90°, \cos 90°, \cos 0°$, *i.e.*, 0, 0, 1.

2. Let foot of perpendicular on X-axis from the point $(3, -5, 12)$ is A.

$\therefore$ Coordinates of the point A = (3, 0, 0)

Then, distance of the point (3, –5, 12) from X-axis

= distance between the points (3, –5, 12) and (3, 0, 0)

$= \sqrt{(3-3)^2 + (0+5)^2 + (0-12)^2}$

$= \sqrt{0 + 25 + 144} = \sqrt{169} = 13$ units

TR!CK

The distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

3. Let $A = (-2, 4, -5)$ and $B = (1, 2, 3)$

Then, $AB = \sqrt{(1+2)^2 + (2-4)^2 + (3+5)^2}$

$= \sqrt{9 + 4 + 64} = \sqrt{77}$ units.

TR!CK

For the line joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2), direction cosines are,

$$\frac{x_2 - x_1}{AB}, \frac{y_2 - y_1}{AB}, \frac{z_2 - z_1}{AB}$$

where, $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Therefore, the direction cosines of line joining the given points are:

$$\frac{1-(-2)}{\sqrt{77}}, \frac{2-4}{\sqrt{77}}, \frac{3-(-5)}{\sqrt{77}}$$

i.e., $$\frac{3}{\sqrt{77}}, \frac{-2}{\sqrt{77}}, \frac{8}{\sqrt{77}}$$

4. Vector equation of a line passing through the point $\vec{a} = 3\hat{i} + 4\hat{j} + 5\hat{k}$ and parallel to vector $\vec{b} = 2\hat{i} + 2\hat{j} - 3\hat{k}$ is $\vec{r} \times \vec{b} = \vec{a} \times \vec{b}$.

$\Rightarrow \vec{r} \times (2\hat{i} + 2\hat{j} - 3\hat{k}) = (3\hat{i} + 4\hat{j} + 5\hat{k}) \times (2\hat{i} + 2\hat{j} - 3\hat{k})$

5. Clearly, the direction ratios of the given line are 3, 5, 6.

Required line is parallel to given line.

$\therefore$ Direction ratios of required line = 3, 5, 6

Therefore, required line passes through the point $(-2, 4, -5)$ and its direction ratios are 3, 5, 6.

TR!CK

Equation of a line passing through point (x_1, y_1, z_1) whose direction ratios are a, b, c, is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

$\therefore$ $$\frac{x-(-2)}{3} = \frac{y-4}{5} = \frac{z-(-5)}{6}$$

or $$\frac{x+2}{3} = \frac{y-4}{5} = \frac{z+5}{6}$$

6. The direction ratios of given lines are 1, 2, 1 and 2, 3, 1 respectively.

TR!CK

The angle between two lines $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ and $\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$ is

$$\cos\theta = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$\therefore \quad \cos\theta = \left| \frac{(1)(2)+(2)(3)+(1)(1)}{\sqrt{(1)^2+(2)^2+(1)^2}\,\sqrt{(2)^2+(3)^2+(1)^2}} \right|$$

$$= \left| \frac{2+6+1}{\sqrt{1+4+1}\sqrt{4+9+1}} \right| = \frac{9}{\sqrt{6}\sqrt{14}} = \frac{9}{2\sqrt{21}}$$

$\therefore$ Required angle is $\theta = \cos^{-1}\left(\frac{9}{2\sqrt{21}}\right)$.

7. Given equation of lines are $\frac{x-5}{7} = \frac{y+2}{-5} = \frac{z}{1}$ and $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$

TiP

If two lines are perpendicular to each other, then the sum of the product of corresponding direction ratios will be zero.

Here, $l_1 = 7, m_1 = -5, n_1 = 1, l_2 = 1, m_2 = 2, n_2 = 3$

Then, $\quad l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

$\therefore \quad (7)(1) + (-5)(2) + (1)(3) = 7 - 10 + 3 = 10 - 10 = 0$

Therefore, lines are perpendicular to each other.

COMMON ERROR

Sometimes students get confused with the condition of perpendicularity.

Short Answer Type-I Questions

1. Direction ratios of the line AB are following

$2-1, 6-2, 3-7$ *i.e.*, 1, 4, –4

and direction ratios of the line BC are following

$3-2, 10-6, -1-3$ *i.e.*, 1, 4, – 4

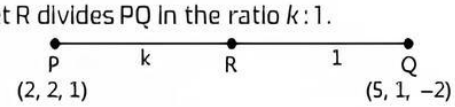
Clearly that, $\frac{1}{1} = \frac{4}{4} = \frac{-4}{-4}$

i.e., direction ratios of the line AB and BC are proportional.

$\therefore$ Lines AB and BC are parallel but point B is common in both lines.

Hence, points A, B and C are collinear. **Hence proved.**

2. Let R divides PQ in the ratio $k:1$.



TR!CK

The coordinates of any point, which divides the join of points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) in the ratio $m:n$ internally are:

$$\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}, \frac{mz_2+nz_1}{m+n}\right)$$

By section formula,

$$R = \left(\frac{5k+2}{k+1}, \frac{k+2}{k+1}, \frac{-2k+1}{k+1}\right)$$

Given that, x-coordinate of $R = 4$

$$\therefore \quad \frac{5k+2}{k+1} = 4$$

$$\Rightarrow \quad 5k+2 = 4k+4 \Rightarrow k = 2$$

So, z-coordinate of $R = \frac{-2k+1}{k+1} = \frac{-2(2)+1}{2+1} \qquad (\because k = 2)$

$$= \frac{-4+1}{3} = \frac{-3}{3} = -1$$

3. Given equation of line is $5x - 3 = 15y + 7 = 3 - 10z$

or $\quad 5\left(x - \frac{3}{5}\right) = 15\left(y + \frac{7}{15}\right) = -10\left(z - \frac{3}{10}\right)$

or $\quad \dfrac{x-\frac{3}{5}}{\frac{1}{5}} = \dfrac{y+\frac{7}{15}}{\frac{1}{15}} = \dfrac{z-\frac{3}{10}}{-\frac{1}{10}}$

or $\quad \dfrac{x-\frac{3}{5}}{6} = \dfrac{y-\left(-\frac{7}{15}\right)}{2} = \dfrac{z-\frac{3}{10}}{-3}$

[$\because$ Multiply denominator by 30]

Here DR's are (6, 2, –3).

TiP

Cartesian equation of a line is

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$$

where, (x_1, y_1, z_1) are the coordinates of the point from which line passes and dr's of the line are $\langle a, b, c \rangle$.

$\therefore$ DC's of a line are

$$\left(\frac{6}{\sqrt{6^2+2^2+(-3)^2}}, \frac{2}{\sqrt{6^2+2^2+(-3)^2}}, \frac{-3}{\sqrt{6^2+2^2+(-3)^2}}\right)$$

or $\left(\frac{6}{\sqrt{49}}, \frac{2}{\sqrt{49}}, \frac{-3}{\sqrt{49}}\right)$ or $\left(\frac{6}{7}, \frac{2}{7}, -\frac{3}{7}\right)$

Thus the coordinates of the required point are $\left(\frac{3}{5}, -\frac{7}{15}, \frac{3}{10}\right)$

4. Given equation of line is,

$$6x - 12 = 3y + 9 = 2z - 2$$

$$\Rightarrow \quad 6(x-2) = 3(y+3) = 2(z-1)$$

$\therefore$ Standard symmetric form of the given line is,

$$\frac{x-2}{1/6} = \frac{y-(-3)}{1/3} = \frac{z-1}{1/2}$$

So, direction ratios of this line are $\left\langle \frac{1}{6}, \frac{1}{3}, \frac{1}{2} \right\rangle$

TiP

Direction ratios of two parallel lines are proportions to each other.

Since, lines are parallel so the direction ratios of the parallel line are

$$\left\langle \frac{k}{6}, \frac{k}{3}, \frac{k}{2} \right\rangle \Rightarrow \langle 1, 2, 3 \rangle \qquad (\text{put } k = 6)$$

Hence, required direction cosines are

$$< \frac{1}{\sqrt{1^2+2^2+3^2}}, \frac{2}{\sqrt{1^2+2^2+3^2}}, \frac{3}{\sqrt{1^2+2^2+3^2}} >$$

or $< \frac{1}{\sqrt{1+4+9}}, \frac{2}{\sqrt{1+4+9}}, \frac{3}{1+4+9} >$

or $< \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} >$.

5. Given line is

$$5x-25 = 14-7y = 35z$$

TR!CK

The vector equation of a line passing through the point with position vector $\vec{a}$ and parallel to a given vector $\vec{b}$ is $\vec{r} = \vec{a} + \lambda\vec{b}$.

$\Rightarrow \quad 5(x-5) = -7(y-2) = 35z$

$\Rightarrow \quad \frac{x-5}{1/5} = \frac{y-2}{-1/7} = \frac{z-0}{1/35}$

$\Rightarrow \quad \frac{x-5}{7} = \frac{y-2}{-5} = \frac{z-0}{1}$

$\therefore$ Direction ratios of this line are 7, −5, 1.

Which is the required cartesian equation of a line.

So, vector equation of the line which passes through the point A (1, 2, −1) and its direction ratios are proportional to 7, −5, 1, is

$$\vec{r} = \hat{i} + 2\hat{j} - \hat{k} + \lambda(7\hat{i} - 5\hat{j} + \hat{k})$$

COMMON ERR!OR

Here some students takes wrong direction ratios and point of the given line because of non-standard form of given line.

Firstly students should convert the given equation of line in standard form then takes the right value of point and direction ratios of the line.

6. The direction ratios of given lines

$\vec{r} = 2\hat{i} - 5\hat{j} + \hat{k} + \lambda(3\hat{i} + 2\hat{j} + 6\hat{k})$

and $\vec{r} = 7\hat{i} - 6\hat{k} + \mu(\hat{i} + 2\hat{j} + 2\hat{k})$

are $\vec{b_1} = 3\hat{i} + 2\hat{j} + 6\hat{k}$ and $\vec{b_2} = \hat{i} + 2\hat{j} + 2\hat{k}$ respectively.

TR!CK

The angle between two lines having direction ratios parallel to $\vec{b_1}$ and $\vec{b_2}$ is $\cos\theta = \frac{|\vec{b_1} \cdot \vec{b}|}{|\vec{b_1}||\vec{b_2}|}$

$\therefore$ The angle between two lines is

$$\cos\theta = \frac{|\vec{b_1}\cdot\vec{b_2}|}{|\vec{b_1}||\vec{b_2}|} = \frac{|3\times1 + 2\times2 + 6\times2|}{\sqrt{(3)^2+(2)^2+(6)^2}\sqrt{(1)^2+(2)^2+(2)^2}}$$

$$= \frac{|3+4+12|}{\sqrt{9+4+36}\sqrt{1+4+4}} = \frac{19}{\sqrt{49}\sqrt{9}} = \frac{19}{7\times3} = \frac{19}{21}$$

$\therefore \quad \theta = \cos^{-1}\left(\frac{19}{21}\right)$

7. Equation of first line is $\frac{x+3}{3} = \frac{y-1}{5} = \frac{z+3}{4} = t$

$\Rightarrow \quad x = 3t-3,\ y = 5t+1,\ z = 4t-3$

$\therefore$ Vector equation of line is

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$= (3t-3)\hat{i} + (5t+1)\hat{j} + (4t-3)\hat{k}$$

$\Rightarrow \quad \vec{r} = (-3\hat{i} + \hat{j} - 3\hat{k}) + t(3\hat{i} + 5\hat{j} + 4\hat{k}) \quad$ ...(1)

Similarly, vector equation of second line is

$$\vec{r} = (s-1)\hat{i} + (s+4)\hat{j} + (2s+5)\hat{k}$$

$\Rightarrow \quad \vec{r} = (-\hat{i} + 4\hat{j} + 5\hat{k}) + s(\hat{i} + \hat{j} + 2\hat{k}) \quad$ ...(2)

From eqs. (1) and (2), the vector equation of lines are parallel to the vectors

$$\vec{b_1} = 3\hat{i} + 5\hat{j} + 4\hat{k}$$

and $\quad \vec{b_2} = \hat{i} + \hat{j} + 2\hat{k}$

TiP

The angle between lines and their various conditions should be learned thoroughly.

$\therefore$ Angle between lines

θ = angle between vectors $\vec{b_1}$ and $\vec{b_2}$

$$= \cos^{-1}\frac{|\vec{b_1}\cdot\vec{b_2}|}{|\vec{b_1}||\vec{b_2}|} = \cos^{-1}\left(\frac{|3\times1+5\times1+4\times2|}{\sqrt{3^2+5^2+4^2}\sqrt{1^2+1^2+2^2}}\right)$$

$$= \cos^{-1}\left(\frac{16}{5\sqrt{2}\times\sqrt{6}}\right) = \cos^{-1}\left(\frac{8}{5\sqrt{3}}\right)$$

8. The standard form of given lines are

$$\frac{x-1}{-3} = \frac{y-2}{2p/7} = \frac{z-3}{2} \text{ and } \frac{x-1}{-3p/7} = \frac{y-5}{1} = \frac{z-6}{-5}$$

TiP

First convert the given equation of lines in standard form i.e., $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$.

If these lines are perpendicular, then

$$(-3)\times\left(\frac{3p}{-7}\right) + \frac{2p}{7}\times1 + 2\times(-5) = 0$$

$\Rightarrow \quad \frac{11p}{7} = 10 \Rightarrow p = \frac{70}{11}$

9. Given equation of lines are

$$\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{3};\ \frac{x}{-3} = \frac{y}{2} = \frac{z}{5}$$

Here Dr's of given lines are

< 1, 2, 3 > and < −3, 2, 5 >

Let $\vec{b_1} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{b_2} = -3\hat{i} + 2\hat{j} + 5\hat{k}$

Now $\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ -3 & 2 & 5 \end{vmatrix}$

$$= \hat{i}(10-6) - \hat{j}(5+9) + \hat{k}(2+6)$$

$$= 4\hat{i} - 14\hat{j} + 8\hat{k}$$

Since, required line is perpendicular to both the given lines. Thus it is parallel to the $\vec{b_1} \times \vec{b_2}$.

$\therefore$ Required equation is

$$\vec{r} = 2\hat{i} + \hat{j} + 3\hat{k} + \lambda(4\hat{i} - 14\hat{j} + 8\hat{k})$$

Short Answer Type-II Questions

1. Given, equation of lines can be written in standard form as

$$\frac{x-1}{-3} = \frac{y-2}{(\lambda/7)} = \frac{z-3}{2} = (r_1) \text{ let} \quad ...(1)$$

and $$\frac{x-1}{(-3\lambda/7)} = \frac{y-5}{1} = \frac{z-6}{-5} = (r_2) \text{ let} \quad ...(2)$$

TR!CK

Two lines $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ *and* $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ *are perpendicular, if* $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

These lines will intersect at right angle, if

$$-3\left(-\frac{3\lambda}{7}\right) + \left(\frac{\lambda}{7}\right)(1) + 2 \times (-5) = 0$$

$$\Rightarrow \quad \frac{9\lambda}{7} + \frac{\lambda}{7} - 10 = 0$$

$$\Rightarrow \quad \frac{10}{7}\lambda = 10 \Rightarrow \lambda = 7$$

which is the required value of λ.

Now, coordinates of any point on line (1) are

$$(-3r_1 + 1, r_1 + 2, 2r_1 + 3)$$

Coordinates of any point on line (2) are

$$(-3r_2 + 1, r_2 + 5, -5r_2 + 6)$$

Clearly, the line will intersect, if

$$(-3r_1 + 1, r_1 + 2, 2r_1 + 3) = (-3r_2 + 1, r_2 + 5, -5r_2 + 6)$$

for some $r_1, r_2 \in R$.

On comparing, we get

$$-3r_1 + 1 = -3r_2 + 1 \Rightarrow r_1 = r_2$$

$$r_1 + 2 = r_2 + 5 \Rightarrow r_1 - r_2 = 3$$

and $$2r_1 + 3 = -5r_2 + 6 \Rightarrow 2r_1 + 5r_2 = 3$$

which is not possible simultaneously for any $r_1, r_2 \in R$.

Hence, the lines are not intersecting.

COMMON ERR(!)R

Sometimes students does not check that the given equations of lines are in standard form or not. They takes wrong values of d.r.'s of the lines and get the wrong value of λ in perpendicularity condition of lines,

e.g., $(3)(3\lambda) + (\lambda)(1) + (2)(5) = 0$

$\Rightarrow 9\lambda + \lambda + 10 = 0 \Rightarrow \lambda = -1$.

2. Equation of given lines are

$$\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7} \quad ...(1)$$

and $$\frac{x-2}{1} = \frac{y-4}{3} = \frac{z-6}{5} \quad ...(2)$$

TiP

Learn the concepts of skew and intersecting lines.

Coordinates of any point on line (1)

$$= (3\lambda - 1, 5\lambda - 3, 7\lambda - 5) \quad ...(3)$$

and coordinates of any point on line (2)

$$= (\mu + 2, 3\mu + 4, 5\mu + 6)$$

If lines intersect, then a point of these lines be common.

So, for some values of λ and μ,

$$3\lambda - 1 = \mu + 2, \quad 5\lambda - 3 = 3\mu + 4,$$

$$7\lambda - 5 = 5\mu + 6$$

$$\Rightarrow \quad 3\lambda - \mu = 3, \quad 5\lambda - 3\mu = 7, \quad 7\lambda - 5\mu = 11$$

Solving the first two equations from the above,

$$\lambda = \frac{1}{2}$$

and $$\mu = \frac{-3}{2}$$

These values of λ and μ satisfies the third equation.

So, given lines intersect.

Put $\lambda = \frac{1}{2}$ in eq. (3).

Coordinates of required point of intersection

$$= \left(\frac{1}{2}, \frac{-1}{2}, \frac{-3}{2}\right)$$

COMMON ERR(!)R

Some students compute the shortest distance to show that the lines are intersecting. Which leads to difficult for finding intersection point.

3. Given equation of line,

$$\frac{x+2}{3} = \frac{y+1}{2} = \frac{z-3}{2} = r \quad \text{(say)}$$

Let $P(3r-2, 2r-1, 2r+3)$ be any point on the given line.

Given that, the distance of P from the point

$$Q(1, 3, 3) = 5 \text{ units.}$$

$$\therefore \sqrt{(1-3r+2)^2 + (3-2r+1)^2 + (3-2r-3)^2} = 5$$

Squaring on both sides we get,

$$(3-3r)^2 + (4-2r)^2 + (-2r)^2 = 25$$

$$\Rightarrow \quad 9 + 9r^2 - 18r + 16 + 4r^2 - 16r + 4r^2 = 25$$

$$\Rightarrow \quad 17r^2 - 34r = 0$$

$$\Rightarrow \quad 17r(r-2) = 0$$

$$\Rightarrow \quad r = 0 \text{ or } r = 2$$

At $r = 0$,

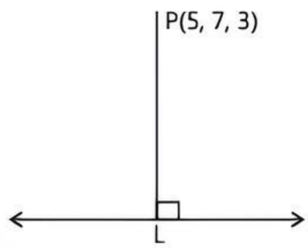
the coordinates of point $P = (-2, -1, 3)$

At $r = 2$,

the coordinates of point $P = (4, 3, 7)$

So, required points are $(-2, -1, 3)$ and $(4, 3, 7)$.

4. Let the foot of perpendicular drawn from point $P(5, 7, 3)$ to the given line be L.



Coordinates of any point on line

$$\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}$$

aer $L(3\lambda+15, 8\lambda+29, -5\lambda+5)$

Now direction ratios of PL are $(3\lambda+15-5, 8\lambda+29-7, -5\lambda+5-3)$

i.e., $(3\lambda+10, 8\lambda+22, -5\lambda+2)$

Since PL is perpendicular to the given line

$\therefore \quad a_1a_2+b_1b_2+c_1c_2=0$

$\Rightarrow \quad 3(3\lambda+10)+8(8\lambda+22)-5(-5\lambda+2)=0$

$\Rightarrow \quad 9\lambda+30+64\lambda+176+25\lambda-10=0$

$\Rightarrow \quad 98\lambda+196=0 \Rightarrow \lambda=-2$

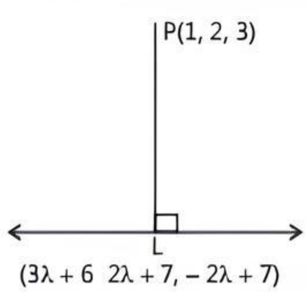
Hence, foot of perpendicular drawn on the given line is $L(3\times-2+15, 8\times-2+29, -5\times(-2)+5)$

i.e., $L(9, 13, 15)$.

5. Let the foot of perpendicular drawn from point P (1, 2, 3) to the given line be L.

TiP

Practice all the different types of questions from the topic of foot of perpendicular.



Coordinates of any point on line $\frac{x-6}{3}=\frac{y-7}{2}=\frac{z-7}{-2}$

are $(3\lambda+6, 2\lambda+7, -2\lambda+7)$.

Let coordinates of point L

$$=(3\lambda+6, 2\lambda+7, -2\lambda+7) \qquad ...(1)$$

$\therefore$ Direction ratios of PL are:

$3\lambda+6-1, 2\lambda+7-2, -2\lambda+7-3$

i.e., $3\lambda+5, 2\lambda+5, -2\lambda+4$

Direction ratios of given line $= 3, 2, -2$

PL is perpendicular to the given line whose direction ratios are $3, 2, -2$.

$\therefore$ From the formula, $a_1a_2+b_1b_2+c_1c_2=0$

$\Rightarrow \quad 3(3\lambda+5)+2(2\lambda+5)+(-2)(-2\lambda+4)=0$

$\Rightarrow \quad 9\lambda+15+4\lambda+10+4\lambda-8=0$

$17\lambda+17=0$

$\Rightarrow \quad 17\lambda=-17 \Rightarrow \lambda=-1$

Put $\lambda=-1$ in eq. (1),

$L=(3, 5, 9)$

$\therefore$ Coordinates of foot of perpendicular $L=(3, 5, 9)$

Length of perpendicular

$$=PL=\sqrt{(3-1)^2+(5-2)^2+(9-3)^2}$$

$$=\sqrt{2^2+3^2+6^2}=\sqrt{4+9+36}$$

$$=\sqrt{49}=7 \text{ units}$$

$\therefore$ Dr's of PL are $\langle -3+5, -2+5, 2+4 \rangle$ i.e., $\langle 2, 3, 6 \rangle$

$\therefore$ Equation of perpendicular PL is:

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{6}$$

COMMON ERROR

Mostly students have an issue with problems involving foot of perpendicular and image.

6.

TR!CK

The shortest distance between the lines

$$\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$$

and $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$ *is:*

$$d=\left|\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}}\right|$$

Given equations, $\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}$

or $\quad \frac{x-(-1)}{7}=\frac{y-(-1)}{-6}=\frac{z-(-1)}{1}$

Here, $x_1=-1, y_1=-1, z_1=-1, a_1=7, b_1=-6, c_1=1$

and $\quad \frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}$

Here, $x_2=3, y_2=5, z_2=7, a_2=1, b_2=-2, c_2=1$

$$\therefore \begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

$$=\begin{vmatrix} 3+1 & 5+1 & 7+1 \\ 7 & -6 & 1 \\ 1 & -2 & 1 \end{vmatrix}=\begin{vmatrix} 4 & 6 & 8 \\ 7 & -6 & 1 \\ 1 & -2 & 1 \end{vmatrix}$$

$$=4(-6+2)-6(7-1)+8(-14+6)$$

$$=-16-36-64$$

$$=-52-64=-116$$

and $\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}$

$$=\sqrt{(-6+2)^2+(1-7)^2+(-14+6)^2}$$

$$=\sqrt{16+36+64}=\sqrt{116}$$

So, required shortest distance

$$d=\left|\frac{-116}{\sqrt{116}}\right|=\sqrt{116}=2\sqrt{29} \text{ units}$$

7.

TiP

Firstly, student should write the equation in standard form $\vec{r}=a_1+\lambda b_1, \vec{r}=a_2+\mu b_2$ then solve the shortest distance.

The given vector equation of lines are

$$\vec{r}=(4\hat{i}-\hat{j})+\lambda(\hat{i}+2\hat{j}-3\hat{k})$$

and $\vec{r}=(\hat{i}-\hat{j}+2\hat{k})+\mu(2\hat{i}+4\hat{j}-5\hat{k})$

Here, $\vec{a_1}=4\hat{i}-\hat{j}, \vec{b_1}=\hat{i}+2\hat{j}-3\hat{k}$

$\vec{a_2}=\hat{i}-\hat{j}+2\hat{k}, \ \vec{b_2}=2\hat{i}+4\hat{j}-5\hat{k}$

$\therefore \ \vec{a_2}-\vec{a_1}=(\hat{i}-\hat{j}+2\hat{k})-(4\hat{i}-\hat{j})=-3\hat{i}+2\hat{k}$

TR!CK

If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$

Then $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$

$$\vec{b_1}\times\vec{b_2}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & -3\\ 2 & 4 & -5\end{vmatrix}$$

$$=(-10+12)\hat{i}-(-5+6)\hat{j}+(4-4)\hat{k}=2\hat{i}-\hat{j}$$

$\therefore |\vec{b_1}\times\vec{b_2}|=\sqrt{(2)^2+(-1)^2}=\sqrt{4+1}=\sqrt{5}$

TR!CKS

If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$.

Then

- $|\vec{a}|=\sqrt{a_1^2+a_2^2+a_3^2}$
- $(\vec{a}\cdot\vec{b})=a_1b_1+a_2b_2+a_3b_3$

where, $\hat{i}\cdot\hat{i}=\hat{j}\cdot\hat{j}=\hat{k}\cdot\hat{k}=1$ and $\hat{i}\cdot\hat{j}=\hat{j}\cdot\hat{k}=\hat{k}\cdot\hat{i}=0$

and $(\vec{a_2}-\vec{a_1})\cdot(\vec{b_1}\times\vec{b_2})=(-3\hat{i}+2\hat{k})\cdot(2\hat{i}-\hat{j})$

$=(-3)\times(2)+0\times(-1)+2\times0=-6$

TR!CK

If the lines are $\vec{r}=\vec{a_1}+\lambda\vec{b_1}$ and $\vec{r}=\vec{a_2}+\mu\vec{b_2}$.

Then, shortest distance, $d=\left|\dfrac{(\vec{b_1}\times\vec{b_2})\cdot(\vec{a_2}-\vec{a_1})}{|\vec{b_1}\times\vec{b_2}|}\right|$

Now, required shortest distance $=\left|\dfrac{(\vec{a_2}-\vec{a_1})\cdot(\vec{b_1}\times\vec{b_2})}{|\vec{b_1}\times\vec{b_2}|}\right|$

$$=\left|\frac{-6}{\sqrt{5}}\right|=\frac{6}{\sqrt{5}}\text{ units.}$$

8. The given vector equation of lines.

$$\vec{r}=3\hat{i}+5\hat{j}+7\hat{k}+\lambda(\hat{i}-2\hat{j}+\hat{k})$$

and $\vec{r}=(-\hat{i}-\hat{j}-\hat{k})+\mu(7\hat{i}-6\hat{j}+\hat{k})$

Here, $\vec{a_1}=3\hat{i}+5\hat{j}+7\hat{k}, \vec{b_1}=\hat{i}-2\hat{j}+\hat{k}$

$\vec{a_2}=-\hat{i}-\hat{j}-\hat{k}, \vec{b_2}=7\hat{i}-6\hat{j}+\hat{k}$

$\therefore \ \vec{a_2}-\vec{a_1}=(-\hat{i}-\hat{j}-\hat{k})-(3\hat{i}+5\hat{j}+7\hat{k})$

$=-4\hat{i}-6\hat{j}-8\hat{k}$

$$\vec{b_1}\times\vec{b_2}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -2 & 1\\ 7 & -6 & 1\end{vmatrix}$$

$=(-2+6)\hat{i}-(1-7)\hat{j}+(-6+14)\hat{k}$

$=4\hat{i}+6\hat{j}+8\hat{k}$

$\therefore \ |\vec{b_1}\times\vec{b_2}|=\sqrt{(4)^2+(6)^2+(8)^2}$

$=\sqrt{16+36+64}=\sqrt{116}$

and $(\vec{a_2}-\vec{a_1})\cdot(\vec{b_1}\times\vec{b_2})$

$=(-4\hat{i}-6\hat{j}-8\hat{k})\cdot(4\hat{i}+6\hat{j}+8\hat{k})$

$=(-4)(4)+(-6)(6)+(-8)(8)$

$=-16-36-64=-116$

Now, required shortest distance $=\left|\dfrac{(\vec{a_2}-\vec{a_1})\cdot(\vec{b_1}\times\vec{b_2})}{|\vec{b_1}\times\vec{b_2}|}\right|$

$$=\left|\frac{-116}{\sqrt{116}}\right|=\frac{116}{\sqrt{116}}=\sqrt{116}=2\sqrt{29}\text{ units.}$$

9. The given lines are passing through the points

$\vec{a_1}=\hat{i}+2\hat{j}-4\hat{k}$ and $\vec{a_2}=3\hat{i}+3\hat{j}-5\hat{k}$

Here, $\vec{b_1}=2\hat{i}+3\hat{j}+6\hat{k}$ and $\vec{b_2}=4\hat{i}+6\hat{j}+12\hat{k}=2\vec{b_1}$

So, both lines are parallel to the vector

$\vec{b}=2\hat{i}+3\hat{j}+6\hat{k}.$

$\therefore \ \vec{a_2}-\vec{a_1}=(3\hat{i}+3\hat{j}-5\hat{k})-(\hat{i}+2\hat{j}-4\hat{k})$

$=2\hat{i}+\hat{j}-\hat{k}$

Now, $(\vec{a_2}-\vec{a_1})\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 1 & -1\\ 2 & 3 & 6\end{vmatrix}$ $[\because \vec{b}=\vec{b_1}]$

$=(6+3)\hat{i}-(12+2)\hat{j}+(6-2)\hat{k}$

$=9\hat{i}-14\hat{j}+4\hat{k}$

and $|\vec{b}|=\sqrt{(2)^2+(3)^2+(6)^2}$

$=\sqrt{4+9+36}=\sqrt{49}=7$

$\therefore$ Required distance between given lines

$$=\frac{|(\vec{a_2}-\vec{a_1})\times\vec{b}|}{|\vec{b}|}$$

$$= \frac{\sqrt{(9)^2 + (-14)^2 + (4)^2}}{7}$$

$$= \sqrt{81 + 196 + 16} = \frac{\sqrt{293}}{7}$$

COMMON ERR!OR

Some students write the wrong values of a_1, a_2 and b_1, b_2 that's why the answer of shortest distance is wrongly calculated.

10. Here, $\vec{a_1} = 0\hat{i} + 0\hat{j} + a\hat{k} = a\hat{k}$.

$\vec{b_1} = m_1\hat{i} + \hat{j} + 0\hat{k} = m_1\hat{i} + \hat{j}$

and $\vec{a_2} = 0\hat{i} + 0\hat{j} + (-a)\hat{k} = -a\hat{k}$,

$\vec{b_2} = m_2\hat{i} + \hat{j} + 0\hat{k} = m_2\hat{i} + \hat{j}$

Now, $\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ m_1 & 1 & 0 \\ m_2 & 1 & 0 \end{vmatrix}$

$= 0\hat{i} + 0\hat{j} + (m_1 - m_2)\hat{k}$

$= (m_1 - m_2)\hat{k}$

and $\vec{a_2} - \vec{a_1} = -a\hat{k} - a\hat{k} = -2a\hat{k}$

Also, $|\vec{b_1} \times \vec{b_2}| = \sqrt{(m_1 - m_2)^2} = (m_1 - m_2)$

$\therefore$ Shortest distance

$$= \frac{|(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})|}{|\vec{b_1} \times \vec{b_2}|} = \frac{|-2a\hat{k} \cdot (m_1 - m_2)\hat{k}|}{(m_1 - m_2)}$$

$$= \frac{|-2a(m_1 - m_2)|}{(m_1 - m_2)} = \frac{2a(m_1 - m_2)}{(m_1 - m_2)} = 2a$$

11. Let the required line be $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$. ...(1)

If line (1) intersects the line $\frac{x-1}{2} = \frac{y+3}{4} = \frac{z-5}{3}$, then

$$\begin{vmatrix} 1-0 & -3-0 & 5-0 \\ a & b & c \\ 2 & 4 & 3 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 1 & -3 & 5 \\ a & b & c \\ 2 & 4 & 3 \end{vmatrix} = 0$$

$\Rightarrow 29a - 7b - 10c = 0$ ...(2)

Similarly, if the line (1) intersects the line

$\frac{x-4}{2} = \frac{y+3}{3} = \frac{z-14}{4}$, then

$$\begin{vmatrix} 4-0 & -3-0 & 14-0 \\ a & b & c \\ 2 & 3 & 4 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 4 & -3 & 14 \\ a & b & c \\ 2 & 3 & 4 \end{vmatrix} = 0$$

$\Rightarrow 54a - 12b - 18c = 0$

$\Rightarrow 9a - 2b - 3c = 0$ ...(3)

From eqs. (2) and (3), we get

$$\frac{a}{1} = \frac{b}{-3} = \frac{c}{5}$$

$\therefore$ The line is $\frac{x}{1} = \frac{y}{-3} = \frac{z}{5}$.

12. The position vectors of arbitrary points on the given lines are as follows

$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda_1(2\hat{i} + 3\hat{j} + 4\hat{k})$

$= (1 + 2\lambda_1)\hat{i} + (2 + 3\lambda_1)\hat{j} + (3 + 4\lambda_1)\hat{k}$

and $\vec{r} = (2\hat{i} + 3\hat{j} + 4\hat{k}) + \lambda_2(3\hat{i} + 4\hat{j} + 5\hat{k})$

$= (2 + 3\lambda_2)\hat{i} + (3 + 4\lambda_2)\hat{j} + (4 + 5\lambda_2)\hat{k}$

If lines intersect, then their one point must be common.

So, for some values of λ_1 and λ_2,

$(1 + 2\lambda_1)\hat{i} + (2 + 3\lambda_1)\hat{j} + (3 + 4\lambda_1)\hat{k}$

$= (2 + 3\lambda_2)\hat{i} + (3 + 4\lambda_2)\hat{j} + (4 + 5\lambda_2)\hat{k}$

$\Rightarrow 1 + 2\lambda_1 = 2 + 3\lambda_2$,

$2 + 3\lambda_1 = 3 + 4\lambda_2$,

$3 + 4\lambda_1 = 4 + 5\lambda_2$

Solving first two equations $2\lambda_1 - 3\lambda_2 = 1$ and $3\lambda_1 - 4\lambda_2 = 1$ from the above, we get

$\lambda_1 = -1$ and $\lambda_2 = -1$

These values of λ_1 and λ_2 satisfy the third equation.

So given lines intersect. **Hence proved.**

Put $\lambda_1 = -1$ in the equation of first line.

$\therefore$ $\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) - (2\hat{i} + 3\hat{j} + 4\hat{k})$

$= -\hat{i} - \hat{j} - \hat{k}$

Therefore, position vector of required point of intersection $= -\hat{i} - \hat{j} - \hat{k}$.

Hence, coordinates of required point of intersection

$= (-1, -1, -1)$

Long Answer Type Questions

1. Given lines are

$$\frac{x-1}{3} = \frac{y+1}{2} = \frac{z-1}{5}; \frac{x+2}{4} = \frac{y-1}{3} = \frac{z+1}{-2}$$

Any point on the given lines are

$P(3\lambda + 1, 2\lambda - 1, 5\lambda + 1)$ and $Q(4k - 2, 3k + 1, -2k - 1)$

Now consider $3\lambda + 1 = 4k - 2$ and $2\lambda - 1 = 3k + 1$

$\Rightarrow 3\lambda - 4k = -3$ and $2\lambda - 3k = 2$

$\Rightarrow \lambda = -17$ and $k = -12$

We have to show that these lines do not intersect. It means we have to show that values of λ and k does not satisfy the third equation.

Consider $5\lambda + 1 = -2k - 1$

Put $\lambda = -17$ and $k = -12$

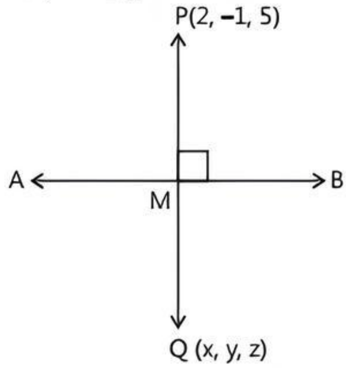
$\therefore$ $5(-17) + 1 = -2(-12) - 1$

$\Rightarrow -85 + 1 = 24 - 1$

$\Rightarrow -84 = 23$, which is not true.

Hence, it shows that given lines do not intersect each other. **Hence proved.**

2. Let $P(2, -1, 5)$ be the given point and AB be the give line $\frac{x-11}{10} = \frac{y+2}{-4} = \frac{z+8}{-11}$



Draw PM perpendicular to AB and produce PM to Q such that M is the mid point of PQ.

Now, any point on the given line is

$$M(10\lambda+11, -4\lambda-2, -11\lambda-8) \quad ...(1)$$

Then Dr's of MP are

$$(10\lambda+11-2, -4\lambda-2+1, -11\lambda-8-5)$$

i.e., $(10\lambda+9, -4\lambda-1, -11\lambda-13)$

Since MP is perpendicular to AB.

$$\therefore \quad a_1a_2+b_1b_2+c_1c_2=0$$

$$\Rightarrow \quad 10(10\lambda+9)-4(-4\lambda-1)-11(-11\lambda-13)=0$$

$$\Rightarrow \quad 100\lambda+90+16\lambda+4+121\lambda+143=0$$

$$\Rightarrow \quad 237\lambda+237=0 \Rightarrow \lambda=-1$$

Put $\lambda=-1$ in eq. (1)

$\therefore$ The coordinate of $M(-10+11, 4-2, 11-8)$

i.e., $M(1, 2, 3)$

Since, M is the mid point of PQ.

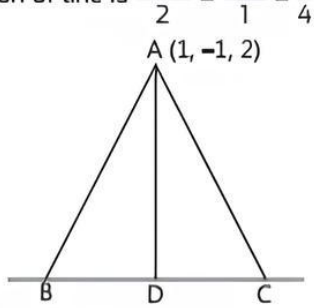
$$\therefore \quad (1, 2, 3) = \left(\frac{2+x}{2}, \frac{-1+y}{2}, \frac{5+z}{2}\right)$$

$$\Rightarrow \quad 1=\frac{2+x}{2}, 2=\frac{-1+y}{2}, 3=\frac{5+z}{2}$$

$$\Rightarrow \quad x=0, y=5, z=1$$

Hence required image of P is $Q(0, 5, 1)$.

3. Given equation of line is $\frac{x+2}{2} = \frac{y-1}{1} = \frac{z}{4}$



Draw AD perpendicular to BC

Since, points B and C lies on given line. Therefore Dr's of line segment BC is $< 2, 1, 4 >$

Now, any point on the line is $D(2\lambda-2, \lambda+1, 4\lambda)$

Now, Dr's of AD are $(2\lambda-2-1, \lambda+1+1, 4\lambda-2)$

i.e., $(2\lambda-3, \lambda+2, 4\lambda-2)$

TR!CK

If two lines are perpendicular then

$a_1a_2+b_1b_2+c_1c_2=0$

Since $AD \perp BC$

$$\therefore \quad (2\lambda-3)2+(\lambda+2)1+(4\lambda-2)4=0$$

$$\Rightarrow \quad 4\lambda-6+\lambda+2+16\lambda-8=0$$

$$\Rightarrow \quad 21\lambda-12=0 \Rightarrow \lambda=\frac{4}{7}$$

$\therefore$ The coordinates of D are $\left(2\times\frac{4}{7}-2, \frac{4}{7}+1, 4\times\frac{4}{7}\right)$

i.e., $\left(-\frac{6}{7}, \frac{11}{7}, \frac{16}{7}\right)$

Now distance $AD = \sqrt{\left(1+\frac{6}{7}\right)^2+\left(-1-\frac{11}{7}\right)^2+\left(2-\frac{16}{7}\right)^2}$

$$= \sqrt{\left(\frac{13}{7}\right)^2+\left(-\frac{18}{7}\right)^2+\left(-\frac{2}{7}\right)^2}$$

$$= \frac{1}{7}\sqrt{169+324+4} = \frac{1}{7}\sqrt{497}$$

$\therefore$ Area of triangle, $ABC = \frac{1}{2}\times BC\times AD = \frac{1}{2}\times 5\times AD$

$[\because BC = 5 \text{ units}]$

$$= \frac{1}{2}\times 5\times\frac{1}{7}\sqrt{497} = \frac{5\sqrt{497}}{14} \text{ Sq. units}$$

4. Any line through (1, 1, 1) can be written as

$$\frac{x-1}{a} = \frac{y-1}{b} = \frac{z-1}{c} \quad ...(1)$$

TiP

The equation of a line passing through a point (x_1, y_1, z_1) and having d.r.'s a, b and c is $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$.

where a, b and c are the direction ratios of line (1).

Now, the line (1) is perpendicular to the lines

$$\frac{x+2}{1} = \frac{y-3}{2} = \frac{z+1}{4}$$

and $$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$$

The direction ratios of the above lines are $< 1, 2, 4 > = < a_1, b_1, c_1 >$ and $< 2, 3, 4 > = < a_2, b_2, c_2 >$ respectively, which are perpendicular to the eq. (1).

TR!CK

Two lines $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ *and* $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ *are said to be perpendicular, if* $a_1a_2+b_1b_2+c_1c_2=0$.

$$\therefore \quad a+2b+4c=0 \quad ...(2)$$

and $$2a+3b+4c=0 \quad ...(3)$$

By cross multiplication, we get

$$\frac{a}{8-12} = \frac{b}{8-4} = \frac{c}{3-4}$$

$$\Rightarrow \quad \frac{a}{-4}=\frac{b}{4}=\frac{c}{-1}=\lambda \text{ (say)}$$

$$\Rightarrow \quad a=-4\lambda,$$

$$b=4\lambda \quad \text{and} \quad c=-\lambda$$

The equation of required line in cartesian form is

$$\frac{x-1}{-4\lambda}=\frac{y-1}{4\lambda}=\frac{z-1}{-\lambda}$$

$$\Rightarrow \quad \frac{x-1}{-4}=\frac{y-1}{4}=\frac{z-1}{-1}$$

TiP

The vector equation of a line passing through a point with position vector $\vec{a}$ and parallel to a given vector $\vec{b}$ is $\vec{r}=\vec{a}+\lambda\vec{b}$.

and in vector form is

$$\vec{r}=(\hat{i}+\hat{j}+\hat{k})+\lambda(-4\hat{i}+4\hat{j}-\hat{k})$$

Let the angle between given lines be θ

TiP

Angle between the lines $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$ is given by

$$\cos\theta=\left|\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right|$$

$$\therefore \quad \cos\theta=\left|\frac{(1)(2)+(2)(3)+(4)(4)}{\sqrt{(1)^2+(2)^2+(4)^2}\sqrt{(2)^2+(3)^2+(4)^2}}\right|$$

$$=\left|\frac{2+6+16}{\sqrt{1+4+16}\sqrt{4+9+16}}\right|$$

$$=\left|\frac{24}{\sqrt{21}\sqrt{29}}\right|=\frac{24}{\sqrt{609}}$$

$$\Rightarrow \quad \theta=\cos^{-1}\left(\frac{24}{\sqrt{609}}\right)$$

COMMON ERROR

Most often students get wrong with the conditions for perpendicularity.

5. The equations of motion of a rocket are:

$$x=2t, \quad y=-4t, \quad z=4t$$

Eliminating 't' between the equations, we obtain the equation of the path

$$\frac{x}{2}=\frac{y}{-4}=\frac{z}{4} \Rightarrow \frac{x-0}{2}=\frac{y-0}{-4}=\frac{z-0}{4}$$

which are the equations of the line passing through the origin having direction ratios $<2,-4,4>$.

This line is the path of the rocket.

At $t=10$,

$$x=2\times10=20,\ y=-4\times10=-40,\ z=4\times10=40$$

$\therefore$ When $t=10$ seconds, the rocket will be at the point $(20,-40,40)$.

Hence, the required distance from the origin at

$$10 \text{ seconds}=\sqrt{(20-0)^2+(-40-0)^2+(40-0)^2}$$

$$=\sqrt{400+1600+1600}=\sqrt{3600}=60 \text{ km}$$

Also, given equation of line,

$$\vec{r}=20\hat{i}-10\hat{j}+40\hat{k}+\mu(10\hat{i}-20\hat{j}+10\hat{k})=\vec{a}_1+\mu\vec{b}$$

and $\vec{a}_2=20\hat{i}-40\hat{j}+40\hat{k}$

$\therefore$ The distance of the point $(20,-40,40)$ from the given line $=\frac{|(\vec{a}_2-\vec{a}_1)\times\vec{b}|}{|\vec{b}|}$

$$=\frac{|(20\hat{i}-40\hat{j}+40\hat{k}-20\hat{i}+10\hat{j}-40\hat{k})\times\vec{b}|}{|\vec{b}|}$$

$$=\frac{|-30\hat{j}\times(10\hat{i}-20\hat{j}+10\hat{k})|}{|10\hat{i}-20\hat{j}+10\hat{k}|}$$

$$=\frac{|-300(\hat{j}\times\hat{i})+0-300(\hat{j}\times\hat{k})|}{\sqrt{(10)^2+(-20)^2+(10)^2}}$$

$$=\frac{|-300(-\hat{k})-300(\hat{i})|}{\sqrt{100+400+100}}=\frac{|300\hat{k}-300\hat{i}|}{\sqrt{600}}$$

$$[\because \hat{j}\times\hat{j}=0, \because \hat{j}\times\hat{i}=-\hat{k},\ \hat{j}\times\hat{k}=\hat{i}]$$

$$=\frac{300\sqrt{(1)^2+(-1)^2}}{\sqrt{600}}=\frac{300\sqrt{2}}{\sqrt{300}\sqrt{2}}=\sqrt{300}$$

$$=10\sqrt{3} \text{ km.}$$

6. Given equation of lines,

$$\vec{r}=6\hat{i}+2\hat{j}+2\hat{k}+\lambda(\hat{i}-2\hat{j}+2\hat{k})=\vec{a}_1+\lambda\vec{b}_1 \quad \text{(say) ...(1)}$$

and $\vec{r}=-4\hat{i}-\hat{k}+\mu(3\hat{i}-2\hat{j}-2\hat{k})=\vec{a}_2+\lambda\vec{b}_2$ (say) ...(2)

Here, $\vec{b}_1\neq\vec{b}_2$

So, the given lines are non-parallel lines *i.e.*, skew lines. There is a unique line-segment AB (A lying on one line and B on the other line), which is at right angles to both the lines. AB is the shortest distance between the lines.

Thus, the shortest possible distance between the insects $=AB$

The position vector of A lying on the line (1) is $(6+\lambda)\hat{i}+(2-2\lambda)\hat{j}+(2+2\lambda)\hat{k}$ for some λ.

The position vector of B lying on the line (2) is $(-4+3\mu)\hat{i}+(-2\mu)\hat{j}+(-1-2\mu)\hat{k}$ for some μ.

Now, $\vec{AB}=\vec{OB}-\vec{OA}$

$$=(-4+3\mu-6-\lambda)\hat{i}+(-2\mu-2+2\lambda)\hat{j}+(-1-2\mu-2-2\lambda)\hat{k}$$

$$=(-10+3\mu-\lambda)\hat{i}+(-2-2\mu+2\lambda)\hat{j}+(-3-2\mu-2\lambda)\hat{k}$$

Since, $\vec{AB}$ is perpendicular to both the lines

$$\therefore \quad (-10+3\mu-\lambda)(1)+(-2-2\mu+2\lambda)(-2) + (-3-2\mu-2\lambda)(2)=0$$

$$\Rightarrow \quad -10+3\mu-\lambda+4+4\mu-4\lambda-6-4\mu-4\lambda=0$$

$$\Rightarrow \quad 3\mu-9\lambda-12=0$$

$$\Rightarrow \quad \mu-3\lambda=4 \quad \text{...(3)}$$

and $(-10+3\mu-\lambda)(3)+(-2-2\mu+2\lambda)(-2) + (-3-2\mu-2\lambda)(-2)=0$

$$\Rightarrow -30+9\mu-3\lambda+4+4\mu-4\lambda+6+4\mu+4\lambda=0$$

$$\Rightarrow \quad 17\mu-3\lambda=20 \quad \text{...(4)}$$

On solving eqs. (3) and (4), we get

$$17(4+3\lambda)-3\lambda=20$$

$$\Rightarrow \quad 68+51\lambda-3\lambda=20$$

$$\Rightarrow \quad 48\lambda=-48 \quad \Rightarrow \quad \lambda=-1$$

put the value of λ in eq. (3), we get

$$\mu-3(-1)=4 \quad \Rightarrow \quad \mu=1$$

The position vector of the points, at which they should be reach so that the distance between them is the shortest, are

$$\vec{OA}=(6-1)\hat{i}+(2+2)\hat{j}+(2-2)\hat{k}=5\hat{i}+4\hat{j} \quad \text{and}$$

$$\vec{OB}=(-4+3)\hat{i}+(-2)\hat{j}+(-1-2)\hat{k}=-\hat{i}-2\hat{j}-3\hat{k}$$

$$\therefore \quad \vec{AB}=(-10+3+1)\hat{i}+(-2-2-2)\hat{j}+(-3-2+2)\hat{k}$$

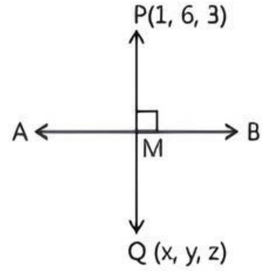
$$=-6\hat{i}-6\hat{j}-3\hat{k}$$

Hence, the shortest distance $=|\vec{AB}|=|-6\hat{i}-6\hat{j}-3\hat{k}|$

$=\sqrt{(-6)^2+(-6)^2+(-3)^2}=\sqrt{36+36+9}=\sqrt{81}=9.$

7. Given vector equation is written in Cartesian form as

$$\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}$$



Let $P(1, 6, 3)$ be the given point and AB be the given line.

Draw PM perpendicular to AB and produce PM to Q such that M is the mid point of PQ.

Now any point on the given line is $M(\lambda, 2\lambda+1, 3\lambda+2)$ ...(1)

Then Dr's of MP are $(\lambda-1, 2\lambda+1-6, 3\lambda+2-3)$

i.e., $(\lambda-1, 2\lambda-5, 3\lambda-1)$

Since, MP is perpendicular to AB.

$$\therefore \quad a_1a_2+b_1b_2+c_1c_2=0$$

$$\Rightarrow \quad 1(\lambda-1)+2(2\lambda-5)+3(3\lambda-1)=0$$

$$\Rightarrow \quad \lambda-1+4\lambda-10+9\lambda-3=0$$

$$\Rightarrow \quad 14\lambda-14=0 \quad \Rightarrow \quad \lambda=1$$

So, coordinates of M are $(1, 2(1)+1, 3(1)+2)$

i.e., $M(1, 3, 5)$

Since, M is the mid point of PQ.

$$\therefore \quad (1, 3, 5)=\left(\frac{1+x}{2}, \frac{6+y}{2}, \frac{3+z}{2}\right)$$

$$\Rightarrow \quad 1=\frac{1+x}{2},\ 3=\frac{6+y}{2},\ 5=\frac{3+z}{2}$$

$$\Rightarrow \quad x=1, y=0, z=7$$

Hence image of $P(1, 6, 3)$ with respect to the given line is $Q(1, 0, 7)$

When we draw a perpendicular line from Q to the Y-axis, then the coordinate of Y-axis is (0, 0, 0).

Now, the distance from Q to the Y-axis

$$=\sqrt{(1-0)^2+0^2+(7-0)^2}=\sqrt{1+49}=\sqrt{50}=5\sqrt{2}$$

8. Given equation of lines are

$$\vec{r}=\lambda(\hat{i}-\hat{j}+\hat{k})=a_1+\lambda\vec{b_1} \quad \text{(say) ...(1)}$$

$$\vec{r}=\hat{i}-\hat{j}+\mu(-2\hat{j}+\hat{k})=a_2+\lambda\vec{b_2} \quad \text{(say) ...(2)}$$

Here $\vec{b_1}\neq\vec{b_2}$

So, the given lines are non-parallel lines i.e., skew lines. There is a unique line segment AB (where A lies on first line and B lies on second line), which is at right angles to both the lines.

Thus the shortest possible distance between the aeroplane flying $= AB$

The position vector of A lying on the line (1) is

$$(0+\lambda)\hat{i}+(0-\lambda)\hat{j}+(0+\lambda)\hat{k}$$

And the position vector of B lying on the line (2) is

$$(1-0\mu)\hat{i}+(-1-2\mu)\hat{j}+(0+\mu)\hat{k}$$

Now, $\vec{AB}=\vec{OB}-\vec{OA}$

$$=(1-\lambda)\hat{i}+(-1-2\mu+\lambda)\hat{j}+(\mu-\lambda)\hat{k}$$

Since $\vec{AB}$ is perpendicular to both the lines.

$$\therefore \quad (1-\lambda)(1)+(-1-2\mu+\lambda)(-1)+(\mu-\lambda)(1)=0$$

and $(1-\lambda)(0)+(-1-2\mu+\lambda)(-2)+(\mu-\lambda)(1)=0$

$$\Rightarrow \quad 1-\lambda+1+2\mu-\lambda+\mu-\lambda=0$$

and $\quad 0+2+4\mu-2\lambda+\mu-\lambda=0$

$$-3\lambda+3\mu+2=0 \quad \text{and} \quad -3\lambda+5\mu+2=0$$

$$\Rightarrow \quad \lambda=\frac{2}{3} \text{ and } \mu=0$$

The position vector of the points at which they should be reach so that the distance between them is the shortest, are

$$\vec{OA}=\frac{2}{3}\hat{i}-\frac{2}{3}\hat{j}+\frac{2}{3}\hat{k} \quad \text{and} \quad \vec{OB}=\hat{i}-\hat{j}$$

Hence, the points on the lines in which the distance between them is shortest, are $\left(\frac{2}{3}\hat{i}-\frac{2}{3}\hat{j}+\frac{2}{3}\hat{k}\right)$ and $(\hat{i}-\hat{j})$.

Now, $\vec{AB}=\vec{OB}-\vec{OA}$

$$=(\hat{i}-\hat{j})-\left(\frac{2}{3}\hat{i}-\frac{2}{3}\hat{j}+\frac{2}{3}\hat{k}\right)=\frac{1}{3}\hat{i}-\frac{1}{3}\hat{j}-\frac{2}{3}\hat{k}$$

$\therefore$ The shortest distance $=|\vec{AB}|$

$$=\sqrt{\left(\frac{1}{3}\right)^2+\left(-\frac{1}{3}\right)^2+\left(-\frac{2}{3}\right)^2}=\sqrt{\frac{1}{9}+\frac{1}{9}+\frac{4}{9}}=\sqrt{\frac{6}{9}}=\sqrt{\frac{2}{3}}$$

Chapter Test

Multiple Choice Questions

Q 1. For which value of λ, the points A (2, 3, −4), B (1, −2, 3) and C (λ, 8, −11) are collinear?

a. 1 b. 2 c. 0 d. 3

Q 2. The equation of line passing through the point (3, 2, 1) and parallel to line $\frac{x-4}{3}=\frac{y+1}{-2}=\frac{z+10}{6}$ is:

a. $\frac{x-3}{3}=\frac{y-2}{-2}=\frac{z-1}{6}$ b. $\frac{x-3}{3}=\frac{y-2}{2}=\frac{z-1}{1}$

c. $\frac{x-4}{3}=\frac{y+1}{-2}=\frac{z+10}{1}$ d. None of these

Assertion and Reason Type Questions

Directions (Q. Nos. 3-4): *In the following questions, each question contains Assertion (A) and Reason (R). Each question has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are:*

a. Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation for Assertion (A)

b. Both Assertion (A) and Reason (R) are true but Reason (R) is not the correct explanation of Assertion (A)

c. Assertion (A) is true but Reason (R) is false

d. Assertion (A) is false and Reason (R) is true

Q 3. Assertion (A): The cartesian equation of a line passing through origin and having direction ratios (a, b, c) is $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$.

Reason (R): The cartesian equation of a line passing through a point (x_1, y_1, z_1) and having direction ratios a, b and c is:

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}.$$

Q 4. Assertion (A): If the direction ratios of two lines are a_1, b_1, c_1 and a_2, b_2, c_2 then the angle θ between them is

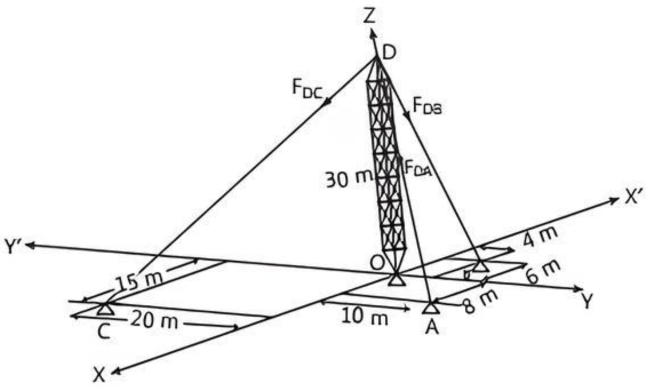
$$\cos\theta=\frac{|a_1a_2+b_1b_2+c_1c_2|}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}.$$

Reason (R): The value of k, so that the lines $\frac{2-x}{-2}=\frac{5y-10}{3k}=\frac{z-3}{4}$ and $\frac{3-3x}{2k}=\frac{y-6}{5}=\frac{9-z}{5}$ intersect at right angle, is 12.

Case Study Based Questions

Q 5. Case Study 1

Consider the following diagram, where the forces in the cable are given.



Based on the above information, solve the following questions:

(i) Find the equation of line along the cable AD.

(ii) Find the length of the cable DC.

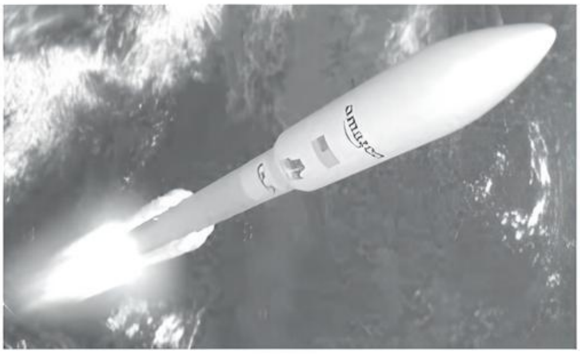
(iii) Find the sum of vectors along the cables.

Or

Find the sum of distances of points A, B and C from the origin *i.e.,* OA + OB + OC.

Q 6. Case Study 2

The equation of motion of a rocket are: $x = 2t$, $y = -4t$, $z = 4t$, where the time 't' is given in seconds and the distance measured is in kilometres.



Based on the above information, solve the following questions:

(i) What is the path of the rocket?

(ii) If another rocket passes through the point (1, −1, 1) and parallel to the previous rocket, then find the equation of the path of another rocket.

(iii) At what distance will the rocket be from the starting point (0, 0, 0) in 20 sec?

Or

At what distance will the rocket be from the following line in 20 seconds?

$$\vec{r}=40\hat{i}-20\hat{j}+80\hat{k}+\lambda(20\hat{i}-40\hat{j}+20\hat{k})$$

Very Short Answer Type Questions

Q 7. Find the equation of a line passing through the point $(1, 0, -1)$ and equally inclined to the axes.

Q 8. Find the angle between the straight lines $\frac{x+1}{2}=\frac{y-2}{5}=\frac{z+3}{4}$ and $\frac{x-1}{1}=\frac{y+2}{2}=\frac{z-3}{-3}$.

Short Answer Type-I Questions

Q 9. A line passes through the point with position vector $2\hat{i}-3\hat{j}+4\hat{k}$ and makes angles 60°, 120° and 45° with X, Y and Z-axes, respectively. Find the equation of the line in the cartesian form.

Q 10. Find the cartesian equation of a line passing through point $(1, 2, -1)$ and parallel to vector $3\hat{i}+2\hat{j}-8\hat{k}$.

Short Answer Type-II Questions

Q 11. The points A $(1, 2, 3)$, B $(-1, -2, -1)$ and C $(2, 3, 2)$ are three vertices of a parallelogram ABCD. Find the equation of CD.

Q 12. Find the coordinates of a point on the line $\frac{x+2}{3}=\frac{y+1}{2}=\frac{z-3}{2}$ at a distance of $\frac{6}{\sqrt{2}}$ from the point $(1, 2, 3)$.

Long Answer Type Questions

Q 13. Find the shortest distance between the lines, whose vector equations are:

$$\vec{r}=(\hat{i}+2\hat{j}+3\hat{k})+\lambda(2\hat{i}+3\hat{j}+4\hat{k})$$

and $\vec{r}=(2\hat{i}+4\hat{j}+5\hat{k})+\mu(4\hat{i}+6\hat{j}+8\hat{k})$

Q 14. Find the foot of perpendicular from P$(1, 2, -3)$ to the line $\frac{x+1}{2}=\frac{y-3}{-2}=\frac{z}{-1}$. Also, find the image of P in the given line.